Distribution Rules Under Dichotomous Preferences: Two Out of Three Ain't Bad

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(with Felix Brandt, Dominik Peters, and Christian Stricker)

Resource Allocation

How to allocate

- a divisible and homogeneous resource
- to several public goods

based on the preferences of multiple agents?

Examples:

- randomizing over deterministic outcomes
- making repeated decisions
- time-sharing
- dividing a budget
- coordinating donations

Studied by Bogomolnaia, Moulin, and Stong (2005) for dichotomous preferences over goods

The Model

A a set of m projects

- $\Delta(n) = \{\delta \in \mathbb{R}^m_+ \colon \sum_{x \in A} \delta(x) = n\}$ distributions over A summing to n

N a set of n agents

- $\emptyset \neq A_i \subset A$ the approval set of agent i
- $u_i(\delta) = \sum_{x \in A_i} \delta(x)$ utility of i for $\delta \in \Delta(n)$
- $\mathcal{A} = (A_1, \ldots, A_n)$ a profile

f a distribution rule mapping a profile $\mathcal{A}$ to a distribution $f(\mathcal{A}) \in \Delta(n)$

Axioms for Distribution Rules

Efficiency (ex-ante Pareto):

there is no $\delta' \in \Delta(n)$ with $u_i(\delta') \geq u_i(\delta)$ for all i and $>$ for some i

Strategyproofness: reporting the approval set truthfully is a weakly dominant strategy

$$u_i(f(A_1, \ldots, A_i, \ldots, A_n)) \geq u_i(f(A_1, \ldots, A_i', \ldots, A_n)) \text{ for all } A_i'$$

Positive share: every agent gets positive utility

$$u_i(\delta) > 0 \text{ for all } i$$

Distribution Rules

Utilitarian rule: uniform distribution over most often approved projects

$$UTIL(\mathcal{A}) = \sum_{x \in A^{\max}} \frac{n}{|A^{\max}|} \cdot x$$

Conditional utilitarian rule (Duddy, 2015; Aziz et al., 2019): agent i distributes 1 uniformly over most often approved projects in A_i

$$CUT(\mathcal{A}) = \sum_{i \in N} \sum_{x \in A_i^{\max}} \frac{1}{|A_i^{\max}|} \cdot x$$

Nash product rule: maximize product of the agents' utilities

$$NASH(\mathcal{A}) = \arg\max_{\delta \in \Delta(n)} \prod_{i \in N} u_i(\delta)$$

Distribution Rules

UTIL

	a	b	c	d
1				
2				
3				
4				
5				
δ	5			

CUT

	a	b	c	d
1	1			
2	1			
3		.5	.5	
4		.5		.5
5	1			
δ	3	1	.5	.5

NASH

	a	b	c	d
1				
2				
3				
4				
5				
δ	3	2		

$u_3(UTIL(\mathcal{A})) = 0 \rightarrow UTIL$ violates positive share

$CUT(\mathcal{A})$ is Pareto dominated by $3.5 \cdot a + 1.5 \cdot b \rightarrow CUT$ is not efficient

$NASH$ is not strategyproof (Aziz et al., 2019)

Two Out of Three ... Ain't Bad

	UTIL	*CUT*	*NASH*
Efficiency	✓	–	✓
Strategyproofness	✓	✓*	–*
Positive share	–	✓	✓

*Aziz et al. (2019)

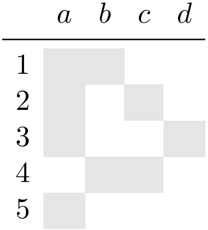
Theorem. No distribution rule satisfies efficiency, strategyproofness, and positive share (for $m \geq 4$, $n \geq 6$)

Conjectured by Bogomolnaia et al. (2005)

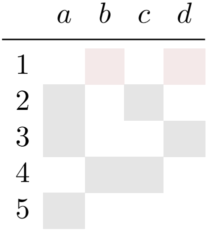
Proof: found by a computer; involves 386 distinct profiles

A Weaker Impossibility

Theorem. No anonymous and neutral distribution rule satisfies efficiency, strategyproofness, and positive share (for $m \geq 4$, $n \geq 5$)



Anonymity and neutrality
$\rightarrow \delta(b) = \delta(c)$
Positive share (for agent 4)
$\rightarrow u_4(\delta) = \delta(b) + \delta(c) > 0$
$\rightarrow u_1(\delta) = \delta(a) + \delta(b) < 5$



Anonymity and neutrality
$\rightarrow \delta'(c) = \delta'(d)$
Efficiency
$\rightarrow \delta'(c) = 0$ or $\delta'(d) = 0$
$\rightarrow u_1(\delta') = \delta'(a) + \delta'(b) = 5$
Strategyproofness
$\rightarrow 5 > u_1(\delta) \geq u_1(\delta') = 5$ ↯

Fairness

Positive share: every agent gets positive utility

$$u_i(\delta) > 0 \text{ for all } i$$

Individual fair share: at least 1 is assigned to projects approved by i

$$u_i(\delta) = \sum_{x \in A_i} \delta(x) \geq 1 \text{ for all } i$$

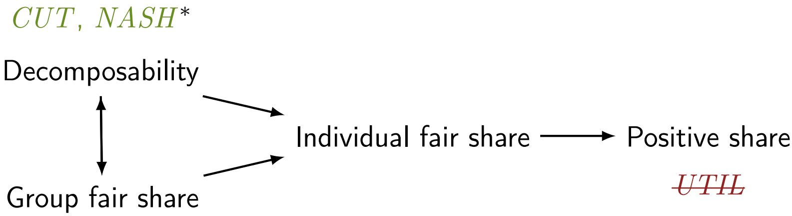
Group fair share: at least $|S|$ is assigned to projects approved by *some* agent in S

$$\sum_{x \in \bigcup_{i \in S} A_i} \delta(x) \geq |S| \text{ for all } S \subset N$$

Decomposability: agent i can assign 1 to projects approved by i

$$\delta = \delta_1 + \cdots + \delta_n \text{ with } \delta_i \in \Delta(1) \text{ and } \mathrm{supp}(\delta_i) \subset A_i$$

Fairness



Proposition. A distribution is decomposable iff it satisfies group fair share

*(Guerdjikova and Nehring, 2014; Brandl et al., 2020)

Participation Incentives

Weak participation: agents weakly prefer participating to abstaining

$$u_i(f(\mathcal{A})) \geq u_i(f(\mathcal{A}_{-i}))$$

Contribution incentive-compatibility: agents weakly prefer participating to abstaining and distributing 1 to approved projects

$$u_i(f(\mathcal{A})) \geq u_i(f(\mathcal{A}_{-i})) + 1$$

Contribution incentive-compatibility ⟶ Weak participation

*CUT, NASH** *UTIL*

*(Brandl et al., 2020)

A New Rule

Sequential utilitarian rule (SUT): sequentially maximize weighted utilitarian welfare

	a	b	c	d	e	w_i		
1	1					1		
2	1					1		
3		1				1	1.5	
4		1				1	1.5	
5	1					1		
6					1	1	1.5	3
	3	2	2	2	1			
		3	2.5	2.5	1.5			
			2.5	2.5	3			

Proposition. SUT is efficient, monotonic, and decomposable

Conclusions

	UTIL	*CUT*	*NASH*	*SUT*	No rule!
Efficiency	✓	–	✓	✓	↯
↳ Decomposable efficiency	✓	✓	✓	✓	
Strategyproofness	✓	✓	–	–	↯
↳ Monotonicity	✓	✓	–	✓	
Decomposability (group fair share)	–	✓	✓	✓	
↳ Positive share	–	✓	✓	✓	↯
Contribution incentive-compatibility	–	✓	✓	–	
↳ Weak participation	✓	✓	✓	–	

Try it at https://dominik-peters.de/demos/portioning