

Consistent Probabilistic Social Choice Revisited

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Brandl et al. (2016) characterized a probabilistic social choice function known as maximal lotteries within a framework based on fractional preference profiles, which abstracts away from individual voters. While this modeling assumption enables a more elegant and transparent proof, it complicates comparison with other results in the literature. The purpose of this note is to transfer their results to the standard model of social choice, where each preference profile is defined for a finite number of voters. Along the way, we prove a slightly stronger version of their main theorem that uses a weaker continuity condition and allows for real-valued (rather than only rational-valued) probabilities.

1. Introduction

Brandl et al. (2016) have shown that a probabilistic social choice function (PSCF) known as maximal lotteries is characterized by population-consistency and composition-consistency. The characterization was proved in a framework that operates on *fractional profiles*, i.e., preference profiles are modeled as functions that associate each possible preference relation with the fraction of voters who share these preferences. While this framework is not without precedent (see, e.g., Young, 1974b, 1975; Young and Levenglick, 1978; Saari, 1995; Myerson, 1995; Dasgupta and Maskin, 2008), it differs from the standard model of social choice theory based on the preferences of individual voters. When PSCFs are defined for fractional profiles, they are *anonymous* (i.e., all voters are treated identically) and *homogeneous* (i.e., replicating the electorate does not affect the outcome). The characterization by Brandl et al. (2016) specifically rests on the underlying framework of fractional profiles when defining *decisiveness* (the set of profiles where the PSCF is resolute is dense in the set of all preference profiles) and *continuity* (the PSCF is upper hemi-continuous).

The purpose of this note is to transfer the theorem by Brandl et al. (2016) to the standard model of social choice, where each preference profile is defined for a finite number of voters. This facilitates comparison with other results in the literature. In addition to

the main theorem by Brandl et al., we also transfer their alternative characterization using Condorcet-consistency and an impossibility result for non-probabilistic pSCFs to the standard setting. These translations are achieved by proving a slightly stronger version of the theorem by Brandl et al. that replaces upper hemi-continuity with a weaker continuity condition. Furthermore, we allow for lotteries with real-valued (rather than only rational-valued) probabilities.

2. Preliminaries

Let U be an infinite universal set of alternatives. The set of *agendas* from which alternatives are to be chosen is the set of finite and non-empty subsets of U , denoted by $\mathcal{F}(U)$. The set of all linear (i.e., complete, transitive, and antisymmetric) *preference relations* over some set $A \in \mathcal{F}(U)$ will be denoted by $\mathcal{L}(A)$. The set of all preference profiles for an agenda A is denoted by $\mathcal{R}_A = \mathbb{N}_0^{\mathcal{L}(A)} \setminus \{0\}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and 0 denotes the empty profile without voters, and $\mathcal{R} = \bigcup_{A \in \mathcal{F}(U)} \mathcal{R}_A$. Hence, preference profiles are *anonymous* in the sense that they only represent how many voters hold each preference relation. For some finite set X , we denote by $\Delta(X)$ the set of all probability distributions over X . We interpret $R(\succ)$ as the number of voters with preference relation $\succ \in \mathcal{L}(A)$. Preference profiles are depicted by tables in which each column represents a preference relation \succ with $R(\succ) > 0$. For $B \subseteq A$ and $R \in \mathcal{R}_A$, $R|_B$ is the restriction of R to alternatives in B , i.e., for all $\succ \in \mathcal{L}(B)$,

$$R|_B(\succ) = \sum_{\succ' \in \mathcal{L}(A): \succ \subseteq \succ'} R(\succ').$$

For all $x, y \in A$, $R(x, y) = R|_{\{x, y\}}(\{(x, y)\})$ is the number of voters who prefer x to y (the set $\{(x, y)\}$ represents the preference relation on two alternatives with $x \succ y$). Elements of $\Delta(A)$ are called *lotteries* and will be written as convex combinations of alternatives. If p is a lottery, p_x is the probability that p assigns to alternative x .

A *probabilistic social choice function (pSCF)* f is a function that, for any agenda $A \in \mathcal{F}(U)$, maps a preference profile $R \in \mathcal{R}_A$ to a non-empty and convex subset of $\Delta(A)$, and satisfies the following four properties.

Homogeneity: For all $R \in \mathcal{R}$ and $k \in \mathbb{N}$, $f(R) = f(kR)$.

Faithfulness: For all $x, y \in U$, $R = \{(1, (x, y))\}$ implies $f(R) \neq \{y\}$.

Continuity: For all $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A$, if there exist $R' \in \mathcal{R}_A$ and a lottery p such that
 $f(R' + kR) = \{p\}$ for all $k \in \mathbb{N}$, then $p \in f(R)$.

Resolvability: For all $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A$, there is $\succ \in \mathcal{L}(A)$ such that $|f(R + \succ)| = 1$.

Let us discuss each of these assumptions individually. Homogeneity is well-known in social choice theory (see, e.g., Smith, 1973; Young, 1977). It demands that replicating the entire electorate does not affect the outcome of the pSCF. In the context of fair division, this property is sometimes called replication invariance.

Faithfulness is an extremely weak condition concerning the special case of a single voter and two alternatives. The original version, due to Young (1974a, 1975), requires that if the voter prefers x to y , x should be selected. Our version is even weaker and only demands that y should not be selected with probability 1. This is the only axiom we impose that interprets the preference relation.

Continuity goes back to Smith (1973) and Young (1975).

Finally, resolvability is a condition concerned with the decisiveness of a pSCF. It demands that whenever a pSCF returns more than one lottery, a single voter can be added such that the pSCF only returns a single lottery for the resulting profile. It was first proposed by Tideman (1987) and picked up more recently by Holliday and Pacuit (2023) and Holliday (2024).

The key differences from the model by Brandl et al. (2016) are replacing upper hemi-continuity with continuity, decisiveness with resolvability, and adding homogeneity.

Probabilistic social choice functions nest traditional (non-probabilistic) set-valued social choice functions. If for each $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A$, $f(R) = \Delta(B)$ for some non-empty $B \subseteq A$, we say that f is *non-probabilistic*. A non-probabilistic pSCF thus returns all lotteries over the alternatives selected by a set-valued social choice function, with which we identify it. This identification behaves well: if a non-probabilistic pSCF satisfies one of our axioms, then the corresponding set-valued social choice function satisfies the analogous axiom in the standard (non-probabilistic) framework.

The central axioms we consider are population-consistency, cloning-consistency, composition-consistency, and Condorcet-consistency. They generalize the corresponding conditions for SCFs, with which they coincide on non-probabilistic pSCFs.

Population-consistency relates choices across varying electorates. More precisely, it requires that whenever a lottery is chosen simultaneously by two electorates, this lottery is also chosen by the union of both electorates. For example, consider the two preference profiles R' and R'' given below.

$\begin{array}{cc} 1 & 1 \\ \hline a & b \\ b & c \\ c & a \end{array}$	$\begin{array}{cc} 1 & 1 \\ \hline a & b \\ c & c \\ b & a \end{array}$	$\begin{array}{ccc} 1 & 1 & 2 \\ \hline a & a & b \\ b & c & c \\ c & b & a \end{array}$	(Example 1)
R'	R''	$R' + R''$	

Population-consistency then demands that any lottery that is chosen in both R' and R'' (say, $1/2 a + 1/2 b$) also has to be chosen when both preference profiles are merged. Formally, a pSCF satisfies population-consistency if for all $A \in \mathcal{F}(U)$, $R', R'' \in \mathcal{R}_A$,

$$f(R') \cap f(R'') \subseteq f(R' + R''). \quad (\text{population-consistency})$$

Population-consistency is arguably one of the most natural axioms for variable electorates and is usually considered in a slightly stronger version, known as *reinforcement* or simply *consistency*, where the inclusion in the equation above is replaced with equality whenever the left-hand side is non-empty. Note that population-consistency is merely

a statement about abstract sets of outcomes, which makes no reference to lotteries whatsoever. It was first considered independently by Smith (1973), Young (1974a), and Fine and Fine (1974) and features prominently in the characterization of scoring rules by Smith (1973) and Young (1975). Population-consistency and its variants have found widespread acceptance in the social choice literature (see, e.g., Young, 1974b; Fishburn, 1978; Young and Levenglick, 1978; Saari, 1990, 1995; Myerson, 1995; Congar and Merlin, 2012; Pivato, 2013; Brandl et al., 2016; Brandl and Peters, 2019, 2022; Nehring and Pivato, 2018; Skowron et al., 2019; Lackner and Skowron, 2021; Lederer, 2024).

Cloning- and composition-consistency prescribe how pSCFs should deal with *decomposable* preference profiles. For two agendas $A, B \in \mathcal{F}(U)$, $B \subseteq A$ is a component in $R \in \mathcal{R}_A$ if the alternatives in B are *adjacent* in all preference relations that appear in R , i.e., for all $a \in A \setminus B$ and $b, b' \in B$, $a \succ b$ if and only if $a \succ b'$ for all $\succ \in \mathcal{L}(A)$ with $R(\succ) > 0$. Intuitively, the alternatives in B can be seen as variants or clones of the same alternative because they have exactly the same relationship to all alternatives that are not in B . For example, consider the following preference profile R in which $B = \{b, b'\}$ constitutes a component.

2	1	3		3	3		2	4		
a	a	b		a	b		b'	b		
b'	b	b'		b	a		b	b'		(Example 2)
b	b'	a								
R				$R _{A'}$			$R _B$			

The ‘essence’ of R is captured by $R|_{A'}$, where $A' = \{a, b\}$ contains only one of the cloned alternatives. It seems reasonable to demand that a pSCF should assign the same probability to a (say, $1/2$) independently of the number of clones of b and the internal relationship between these clones. This condition is called cloning-consistency and was first proposed by Tideman (1987) (see also Zavist and Tideman, 1989). For a formal definition of cloning-consistency, let $A', B \in \mathcal{F}(U)$ and $A = A' \cup B$ such that $A' \cap B = \{b\}$. Then, a pSCF f satisfies cloning-consistency if, for all $R \in \mathcal{R}_A$ such that B is a component in R ,

$$\{(p_x)_{x \in A \setminus B} : p \in f(R)\} = \{(p_x)_{x \in A \setminus B} : p \in f(R|_{A'})\}. \quad (\text{cloning-consistency})$$

On a second look at Example 2, it may appear strange that cloning-consistency does not impose any restrictions on the probabilities that f assigns to the clones. While clones behave completely identically with respect to uncloned alternatives, they are not indistinguishable from *each other*. It seems that the relationships between clones ($R|_B$) should be taken into account as well. For example, one would expect that f assigns more probability to b than to b' because two thirds of the voters prefer b to b' . An elegant and mathematically appealing way to formalize this intuition is to require that the probabilities of the clones b and b' are directly proportional to the probabilities that f assigns to these alternatives when restricting the preference profile to the component $\{b, b'\}$. This condition, known as *composition-consistency*, is due to Laffond et al. (1996)

and was studied in detail for majoritarian SCFs (see, e.g., Laslier, 1996, 1997; Brandt, 2011; Brandt et al., 2011; Horan, 2013) and non-majoritarian SCFs (see, e.g., Brandl et al., 2016; Öztürk, 2020; Berker et al., 2025).

For a formal definition of composition-consistency, let $p \in \Delta(A')$ and $q \in \Delta(B)$ and define

$$(p \times_b q)_x = \begin{cases} p_x & \text{if } x \in A \setminus B, \\ p_b q_x & \text{if } x \in B. \end{cases}$$

The operator \times_b is extended to sets of lotteries $X \subseteq \Delta(A')$ and $Y \subseteq \Delta(B)$ by applying it to all pairs of lotteries in $X \times Y$, i.e., $X \times_b Y = \{p \times_b q \in \Delta(A): p \in X \text{ and } q \in Y\}$.

Then, a pSCF f satisfies composition-consistency if for all $R \in \mathcal{R}_A$ such that B is a component in R ,

$$f(R|_{A'}) \times_b f(R|_B) = f(R). \quad (\text{composition-consistency})$$

In Example 2 above, $1/2a + 1/2b \in f(R|_{A'})$, $2/3b + 1/3b' \in f(R|_B)$, and composition-consistency would imply that $1/2a + 1/3b + 1/6b' \in f(R)$.

As pointed out by Brandl et al. (2016), composition-consistency implies cloning-consistency for both probabilistic and non-probabilistic SCFs.

For $A \in \mathcal{F}(U)$, $R \in \mathcal{R}_A$, $x, y \in A$, the entries $M_R(x, y)$ of the *majority margin matrix* M_R denote the difference between the number of voters who prefer x to y and the number of voters who prefer y to x , i.e.,

$$M_R(x, y) = R(x, y) - R(y, x).$$

An alternative $x \in A$ is a *weak Condorcet winner* for R if $M_R(x, y) \geq 0$ for all $y \in A$, and x is a (strict) *Condorcet winner* for R if $M_R(x, y) > 0$ for all $y \in A \setminus \{x\}$.

A pSCF f is Condorcet-consistent if for all $A \in \mathcal{F}(U)$, $R \in \mathcal{R}_A$

$$f(R) = \{x\} \text{ whenever } x \text{ is a Condorcet winner for } R. \quad (\text{Condorcet-consistency})$$

f is *weakly Condorcet-consistent* if $x \in f(R)$ whenever x is a weak Condorcet winner for R . The continuity of pSCFs implies that every Condorcet-consistent pSCF is also weakly Condorcet-consistent. To see this, let $R \in \mathcal{R}_A$ be a profile with weak Condorcet winner x and $R' \in \mathcal{R}_A$ a profile consisting of a single voter who ranks x first. We then have $f(R' + kR) = \{x\}$ for all $k \in \mathbb{N}$ because x is a Condorcet winner in all these profiles. Hence, continuity implies that $x \in f(R)$.

3. Results

Brandl et al. (2016, Theorem 1) have shown that population-consistency and cloning-consistency are incompatible for non-probabilistic social choice functions in a fractional profile framework.¹ We transfer this statement to the framework of pSCFs via a mapping from one setting to the other (Lemma 2 in Appendix A).

¹They state their theorem for composition-consistency but mention in Footnote 11 that cloning-consistency suffices.

Theorem 1. *There is no non-probabilistic pSCF that satisfies population-consistency and cloning-consistency.*

It is instructive to compare this theorem to other results concerning the compatibility of two types of consistency conditions. Population-consistency and Condorcet-consistency are incompatible (Young and Levenglick, 1978), provided that there are at least nine voters (Brandt et al., 2025). Composition-consistency and Condorcet-consistency can be satisfied simultaneously by non-probabilistic pSCFs. This is, for example, the case for the *essential set*, which returns all alternatives that receive positive probability in a maximal lottery, as well as for several variants of the *uncovered set* and the *Banks set*. Some more discriminating non-probabilistic pSCFs, such as *ranked pairs* and *split cycle*, are Condorcet-consistent and cloning-consistent.

Let us now turn to the main theorem, a characterization of a pSCF called maximal lotteries, proposed by Fishburn (1984).² It was rediscovered and studied extensively in subsequent work (see, e.g., Dutta and Laslier, 1999; Laslier, 2000; Rivest and Shen, 2010; Brandl et al., 2022; Brandl and Brandt, 2020, 2024).³ Maximal lotteries (ML) can be viewed as “probabilistic weak Condorcet winners.” Formally, for all $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A$,

$$ML(R) = \{p \in \Delta(A) : p^\top M_R \geq \mathbf{0}\}. \quad (\text{maximal lotteries})$$

M_R can be interpreted as the payoff matrix of a symmetric zero-sum game and maximal lotteries as the mixed maximin strategies (or Nash equilibrium strategies) of this game. It thus follows from the minimax theorem that $ML(R) \neq \emptyset$ for all $R \in \mathcal{R}$ (von Neumann, 1928). Moreover, $|ML(R)| = 1$ whenever the number of voters in R is odd (Laffond et al., 1997). This implies that ML not only satisfies resolvability, but an even stronger condition: For all $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A$ such that $|ML(R)| > 1$, $|ML(R + \succ)| = 1$ for all $\succ \in \mathcal{L}(A)$. This strong notion of resolvability is violated by virtually all common non-probabilistic pSCFs, including plurality, Borda’s rule, and Nanson’s rule. The other three conditions we demand from pSCFs (homogeneity, faithfulness, and continuity) are easily seen to hold. The proof of the following theorem is deferred to Appendix A.

Theorem 2. *ML is the only pSCF that satisfies population-consistency and composition-consistency.*

Brandl et al. (2016, Remark 5) also discuss an alternative characterization of ML , which can be transferred to the framework of pSCFs.

Theorem 3. *ML is the only pSCF that satisfies population-consistency, cloning-consistency, and weak Condorcet-consistency.*

Note that ML also satisfies composition-consistency (which is stronger than cloning-consistency) and Condorcet-consistency (which is stronger than weak Condorcet-consistency).

²Kreweras (1965) proposed a majoritarian version of maximal lotteries much earlier than Fishburn. This version was independently rediscovered by Felsenthal and Machover (1992), Laffond et al. (1993), and Fisher and Ryan (1995).

³For overviews of the properties of maximal lotteries, the reader is referred to Brandt (2017), Brandl et al. (2022), and Brandt (2026).

Convex-valuedness cannot be dropped from the definition of pSCFs in Theorem 2 and Theorem 3: the function that maps each preference profile R to the set of all maximal lotteries with *rational* probabilities, $ML(R) \cap \mathbb{Q}^A$, satisfies all remaining axioms, including population-consistency, composition-consistency, and Condorcet-consistency, but differs from ML whenever $ML(R)$ is not a singleton. Whether convex-valuedness is required was left open by Brandl et al. (2016, Footnote 18). The example above settles this question in our framework; in the framework of Brandl et al., where all lotteries are rational-valued, it remains open.

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APPENDIX

A. Proofs

Throughout this appendix, we exploit the connection between our model and the fractional-profile framework of Brandl et al. (2016). Every homogeneous pSCF f induces a PSCF F on fractional profiles (Lemma 1) that inherits the consistency properties of f (Lemma 2); Theorem 2 thus reduces to showing that $F = ML$, which lets us reuse much of the machinery of Brandl et al. (2016). We establish the two inclusions separately. For $F \subseteq ML$ (Lemma 6), we reprove the two-alternative case (Lemma 3) from scratch, since our faithfulness and continuity axioms are weaker than the unanimity and upper hemi-continuity assumed by Brandl et al.; for larger agendas, their argument carries over, except that, as we admit real-valued (rather than only rational-valued) probabilities, we first reduce any violation of maximality to one with rational probabilities (Lemma 5). This reduction invokes the fact that population- and composition-consistency imply weak Condorcet-consistency near the uniform profile, which we restate as Lemma 4 (Brandl et al., 2016, Lemma 6). The reverse inclusion $ML \subseteq F$ (Lemma 8) is the only part of the characterization that hinges on continuity: because we weaken upper hemi-continuity, the argument of Brandl et al. (their Lemma 15) no longer applies, and we give a self-contained proof in which the relevant maximal lottery is the *unique* one along a sequence of profiles converging to the given profile. It relies on a construction of skew-symmetric matrices that we recall as Lemma 7 (Brandl et al., 2016, Lemma 14); along the way, we simplify the argument by reducing the case of an even-sized support to that of an odd-sized one.

For a finite set X , let $\Delta_{\mathbb{Q}}(X) = \Delta(X) \cap \mathbb{Q}^X$ be the set of probability distributions with rational values. The set of fractional preference profiles for an agenda $A \in \mathcal{F}(U)$ is $\mathcal{R}_A^* = \Delta_{\mathbb{Q}}(\mathcal{L}(A))$, which can be associated with the $(|A| - 1)$ -dimensional unit simplex in $\mathbb{Q}^{\mathcal{L}(A)}$. Therefore, $R(\succ)$ is the fraction of voters with preference relation $\succ \in \mathcal{L}(A)$. The set of fractional preference profiles is $\mathcal{R}^* = \bigcup_{A \in \mathcal{F}(U)} \mathcal{R}_A^*$.

A probabilistic social choice function F on fractional profiles (PSCF) is a function that, for any agenda $A \in \mathcal{F}(U)$, maps a fractional preference profile $R \in \mathcal{R}_A^*$ to a non-empty and convex subset of $\Delta(A)$, and satisfies the following three properties.

- Faithfulness:* For all $x, y \in U$, $R(\{(x, y)\}) = 1$ implies $F(R) \neq \{y\}$.
Continuity: For all $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A^*$, if there exist $R' \in \mathcal{R}_A^*$ and a lottery p such that

$$F(1/k R' + (1 - 1/k) R) = \{p\}$$
 for all $k \in \mathbb{N}$, then $p \in F(R)$.
Decisiveness: For all $A \in \mathcal{F}(U)$, $\{R \in \mathcal{R}_A^* : |F(R)| = 1\}$ is dense in \mathcal{R}_A^* .

Note that the continuity condition used here is weaker than the upper hemi-continuity condition used by Brandl et al. (2016). As a consequence, we need to reprove some of their statements using this weaker condition.

A PSCF is non-probabilistic if for each $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A^*$, $F(R) = \Delta(B)$ for some non-empty $B \subseteq A$. A PSCF satisfies population-consistency if for all $A \in \mathcal{F}(U)$,

$R', R'' \in \mathcal{R}_A^*$, and $\lambda \in [0, 1] \cap \mathbb{Q}$,

$$F(R') \cap F(R'') \subseteq F(\lambda R' + (1 - \lambda) R''). \quad (\text{population-consistency})$$

A PSCF satisfies composition-consistency, if for all $A', B \in \mathcal{F}(U)$ and $A = A' \cup B$ such that $A' \cap B = \{b\}$, and all $R \in \mathcal{R}_A^*$ such that B is a component in R ,

$$F(R|_{A'}) \times_b F(R|_B) = F(R). \quad (\text{composition-consistency})$$

Moreover, it satisfies cloning-consistency if for all $A', B \in \mathcal{F}(U)$ and $A = A' \cup B$ such that $A' \cap B = \{b\}$, and all $R \in \mathcal{R}_A^*$ such that B is a component in R ,

$$\{(p_x)_{x \in A \setminus B} : p \in F(R)\} = \{(p_x)_{x \in A \setminus B} : p \in F(R|_{A'})\}. \quad (\text{cloning-consistency})$$

Composition-consistency implies cloning-consistency.

Every homogeneous pSCF induces a PSCF. If f is homogeneous, define F as follows: for each $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A^*$, let $n \in \mathbb{N}$ such that $nR \in \mathcal{R}_A$ and let

$$F(R) = f(nR).$$

This construction gives a well-defined PSCF.

Lemma 1. *If F is induced by f , then F is a well-defined PSCF.*

Proof. First, F is well-defined. Let $R \in \mathcal{R}_A^*$ and let $m, n \in \mathbb{N}$ such that $mR, nR \in \mathcal{R}_A$. By homogeneity of f , $f(mR) = f(nmR) = f(mnR) = f(nR)$, and thus $F(R)$ does not depend on the chosen scaling factor.

Second, F is a PSCF. Faithfulness is immediate: if $R(\{(x, y)\}) = 1$, then $R = 1R$ is a non-fractional profile with one voter who prefers x to y , and therefore $f(R) \neq \{y\}$. Convex-valuedness holds because $F(R) = f(nR)$ for some $n \in \mathbb{N}$ and $f(nR)$ is convex. For continuity, fix $R \in \mathcal{R}_A^*$. Suppose there is $R' \in \mathcal{R}_A^*$ such that $F(1/k R' + (1 - 1/k) R) = \{p\}$ for all $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $nR \in \mathcal{R}_A$ and $nR' \in \mathcal{R}_A$. Then, $f(nR' + (k - 1)nR) = \{p\}$ for all $k \in \mathbb{N}$. Because f is continuous, $p \in f(nR) = F(R)$. For decisiveness, fix $R \in \mathcal{R}_A^*$ and choose $n \in \mathbb{N}$ with $nR \in \mathcal{R}_A$. For each $k \in \mathbb{N}$, resolvability of f yields $\succ_k \in \mathcal{L}(A)$ such that $|f(knR + \succ_k)| = 1$. Let

$$R_k = \frac{knR + \succ_k}{kn + 1} = \frac{kn}{kn + 1} R + \frac{1}{kn + 1} \succ_k.$$

Then $R_k \rightarrow R$ as $k \rightarrow \infty$ and $F(R_k) = f(knR + \succ_k)$, so $|F(R_k)| = 1$. Thus $\{R \in \mathcal{R}_A^* : |F(R)| = 1\}$ is dense in \mathcal{R}_A^* . \square

The PSCF induced by the pSCF ML is also denoted by ML . A pSCF that satisfies any of our axioms induces a PSCF that also satisfies that axiom.

Lemma 2. *Let f be a pSCF and denote by F the PSCF induced by f . If f is population-consistent, composition-consistent, cloning-consistent, (weakly) Condorcet-consistent, or non-probabilistic, then so is F .*

Proof. To prove population-consistency, let $R', R'' \in \mathcal{R}_A^*$, $\lambda \in [0, 1] \cap \mathbb{Q}$, and $p \in F(R') \cap F(R'')$. The cases $\lambda \in \{0, 1\}$ are trivial, so assume $\lambda \in (0, 1)$. Choose $m \in \mathbb{N}$ such that $m\lambda, m(1 - \lambda) \in \mathbb{N}$, $m\lambda R' \in \mathcal{R}_A$, and $m(1 - \lambda)R'' \in \mathcal{R}_A$. By homogeneity, $F(R') = f(m\lambda R')$ and $F(R'') = f(m(1 - \lambda)R'')$. Population-consistency of f yields $p \in f(m\lambda R' + m(1 - \lambda)R'')$, and since $m(\lambda R' + (1 - \lambda)R'') = m\lambda R' + m(1 - \lambda)R''$, we obtain $p \in F(\lambda R' + (1 - \lambda)R'')$.

To prove composition-consistency, let $A', B \in \mathcal{F}(U)$ with $A' \cap B = \{b\}$, let $A = A' \cup B$, and let $R \in \mathcal{R}_A^*$ such that B is a component in R . Choose m with $mR \in \mathcal{R}_A$. Then B is also a component in mR , and composition-consistency of f gives

$$f(mR) = f(mR|_{A'}) \times_b f(mR|_B).$$

By homogeneity, $f(mR|_{A'}) = F(R|_{A'})$ and $f(mR|_B) = F(R|_B)$, so $F(R) = F(R|_{A'}) \times_b F(R|_B)$.

The proof of cloning-consistency is similar.

It is immediate that if f is (weakly) Condorcet-consistent or non-probabilistic, then so is F . The proofs are omitted. \square

In the proofs below, we will make use of the fact that cloning-consistency implies *neutrality*, i.e., $\pi(F(R)) = F(\pi(R))$ for all $A \in \mathcal{F}(U)$, $R \in \mathcal{R}_A^*$, and all permutations π of A , where $\pi(R)$ and $\pi(F(R))$ result from relabeling the alternatives according to π . While Brandl et al. (2016, Lemma 1) state this implication for composition-consistency, their proof only uses cloning-consistency.

A.1. $F \subseteq ML$

First, we consider the case of two alternatives.

Lemma 3. *Let F be a PSCF that satisfies population-consistency and composition-consistency. Let $x, y \in U$. Then, for each $R \in \mathcal{R}_{\{x, y\}}^*$, $F(R) = ML(R)$.*

Proof. Let $R \in \mathcal{R}_{\{x, y\}}^*$ with $R(x, y) > R(y, x)$.

Step 1. First, we show that for each $p \in F(R)$, $p(x) \in \{0, 1\}$, that is, $F(R) \subseteq \{x, y\}$. Assume for contradiction that there is $p \in F(R)$ with $p(x) \in (0, 1)$. Composition-consistency and convex-valuedness imply that $\text{int}(\Delta(\{x, y\})) \subseteq F(R)$; this can be extracted from the proof of Lemma 4 of Brandl et al. (2016), which uses continuity only to obtain the two degenerate lotteries. Since F satisfies composition-consistency, it also satisfies neutrality, which implies that $\text{int}(\Delta(\{x, y\})) \subseteq F(R^{x \leftrightarrow y})$, where $R^{x \leftrightarrow y}$ is the profile obtained from R by swapping the labels of x and y . Then, population-consistency implies that $\text{int}(\Delta(\{x, y\})) \subseteq F(R')$ for each $R' \in \mathcal{R}_{\{x, y\}}^*$ with $R(x, y) \geq R'(x, y) \geq R(y, x) = R^{x \leftrightarrow y}(x, y)$. This contradicts decisiveness.

Step 2. Second, we show that $y \notin F(R)$. Assume for contradiction that $y \in F(R)$. If $R' \in \mathcal{R}_{\{x, y\}}^*$ satisfies $R'(x, y) = 0$, then all voters prefer y to x ; by Step 1 (applied to y via neutrality) and convex-valuedness, $F(R')$ is a single degenerate lottery, and since faithfulness yields $F(R') \neq \{x\}$, we obtain $y \in F(R')$. Population-consistency

implies that $y \in F(R')$ for each $R' \in \mathcal{R}_{\{x,y\}}^*$ with $R(x,y) \geq R'(x,y) \geq 0$. By neutrality, $x \in F(R')$ for each $R' \in \mathcal{R}_{\{x,y\}}^*$ with $1 \geq R'(x,y) \geq R(y,x)$. Thus, by convex-valuedness, $\Delta(\{x,y\}) \subseteq F(R')$ for each $R' \in \mathcal{R}_{\{x,y\}}^*$ with $R(x,y) \geq R'(x,y) \geq R(y,x)$. This contradicts Step 1. Together, it follows that $F(R) = \{x\}$.

Neutrality implies that $F(R') = \{y\}$ for each $R' \in \mathcal{R}_{\{x,y\}}^*$ with $R'(y,x) > R'(x,y)$. It follows that $F(R') \subseteq ML(R')$ for each $R' \in \mathcal{R}_{\{x,y\}}^*$.

Step 3. Third, we show that $F(R') = ML(R')$ if $R'(x,y) = R'(y,x)$. Let $\succ \in \mathcal{L}(\{x,y\})$ with $x \succ y$. By the previous two steps, for each $\lambda \in (0,1]$, $F(\lambda \succ + (1-\lambda)R') = \{x\}$. Thus, continuity implies that $x \in F(R')$. By neutrality, $y \in F(R')$. Thus, convex-valuedness implies that $F(R') = \Delta(\{x,y\})$. □

Lemma 4 (Brandl et al., 2016, Lemma 6). *Let F be a PSCF that satisfies population-consistency and composition-consistency. Then, for each $A \in \mathcal{F}(U)$, F satisfies weak Condorcet-consistency in a neighborhood of the uniform profile $\text{uni}(\mathcal{L}(A))$.*

Lemma 5. *Let F be a PSCF that satisfies population-consistency and composition-consistency. If $F \not\subseteq ML$, then there are $A \in \mathcal{F}(U)$, $R \in \mathcal{R}_A^*$, and $p \in F(R) \setminus ML(R)$ with $p \in \Delta_{\mathbb{Q}}(A)$.*

Proof. We start by proving the following claim.

Claim 1. There exist $A \in \mathcal{F}(U)$, $c \in A$, $B \subseteq A \setminus \{c\}$, $\tilde{R} \in \mathcal{R}_A^*$, and $\tilde{p} \in F(\tilde{R})$ such that

- (i) c is a Condorcet winner for \tilde{R} ,
- (ii) $\tilde{p}(c) < 1$,
- (iii) $B \neq \emptyset$ and, for all $x, y \in B$, $\tilde{p}(x) = \tilde{p}(y) > 0$.

Proof of Claim 1. We first record a construction that will be used twice. Let $D \in \mathcal{F}(U)$, $S \in \mathcal{R}_D^*$, $q \in F(S)$, $\hat{a} \in U$, $k \in \mathbb{Z}_+^D \setminus \{0\}$, and $\kappa \in \mathbb{R}_+$. Let $x = \kappa k$ and assume that $x \leq q$ and, if $\hat{a} \in D$, that $x(\hat{a}) = q(\hat{a})$ and $k(\hat{a}) > 0$. Then, there are $\hat{D} = D \cup \{\hat{a}\}$, $\hat{S} \in \mathcal{R}_{\hat{D}}^*$, and $\hat{q} \in F(\hat{S})$ such that

$$\hat{q} = q - x + \|x\| \hat{a}$$

(where q and x assign probability 0 to \hat{a} if $\hat{a} \notin D$), and, for each $z \in D \setminus \{\hat{a}\}$,

$$M_{\hat{S}}(\hat{a}, z) = \sum_{a \in D} \frac{k(a)}{\|k\|} M_S(a, z). \tag{1}$$

Moreover, all majority margins not involving \hat{a} remain unchanged. To see this, choose pairwise disjoint sets C_a of clones of a with $|C_a| = k(a)$ for all $a \in D$ and such that $\hat{a} \in \bigcup_{a \in D} C_a$; if $\hat{a} \in D$, choose $\hat{a} \in C_{\hat{a}}$. Let $C = \bigcup_{a \in D} C_a$. Replace each $a \in D$ by the component $\{a\} \cup C_a$ if $a \neq \hat{a}$ and by the component $C_{\hat{a}}$ if $a = \hat{a}$. By Lemma 3 and composition-consistency, the internal profiles can be chosen so that repeated applications

of composition-consistency yield a profile S^+ and a lottery $q^+ \in F(S^+)$ with $q^+(z) = q(z) - x(z)$ for all $z \in D \setminus \{\hat{a}\}$ and $q^+(z) = \kappa$ for all $z \in C$; moreover, the average majority margin of the alternatives in C_a against a is 0 for each $a \in D \setminus \{\hat{a}\}$. Averaging S^+ over all permutations of C preserves q^+ by neutrality and population-consistency. In the averaged profile, all alternatives in C are clones, and each element of C has, against each $z \in D \setminus \{\hat{a}\}$, the average majority margin in (1). Blowing down C to \hat{a} and applying composition-consistency gives the desired profile \hat{S} and lottery \hat{q} .

Now let $A \in \mathcal{F}(U)$, $R \in \mathcal{R}_A^*$, and $p \in F(R) \setminus ML(R)$. Then, there is $c \in A$ such that

$$\gamma = \sum_{a \in A} p(a)M_R(c, a) > 0.$$

In particular, $p(c) < 1$. Choose $L \in \mathbb{N}$ such that $L\gamma > 2$ and then choose $\eta > 0$ such that $L\eta < 1 - \eta$. Choose $k \in \mathbb{Z}_+^A \setminus \{0\}$ and $\kappa \in \mathbb{R}_+$ such that, for $x = \kappa k$,

$$x \leq p, \quad \|x\| \geq 1 - \eta, \quad \text{and} \quad \sum_{a \in A} \frac{k(a)}{\|k\|} M_R(c, a) > \frac{\gamma}{2}.$$

This is possible by choosing $k/\|k\|$ with support contained in $\text{supp}(p)$ sufficiently close to p and then scaling it down slightly so that $x \leq p$.

Let $\bar{c} \in U \setminus A$. Apply the construction with $D = A$, $S = R$, $q = p$, and $\hat{a} = \bar{c}$. We obtain a profile $\bar{R} \in \mathcal{R}_{A \cup \{\bar{c}\}}^*$ and a lottery $\bar{p} \in F(\bar{R})$ such that

$$\bar{p} = p - x + \|x\| \bar{c} \quad \text{and} \quad M_{\bar{R}}(c, \bar{c}) > \frac{\gamma}{2}.$$

In particular, $\bar{p}(A) \leq \eta$ and $\bar{p}(\bar{c}) \geq 1 - \eta$.

We next replace each alternative $a \in A \setminus \{c\}$ by the mixture of a and \bar{c} that puts relative weights 1 and L on these two alternatives. This can be done by sequentially applying the construction with $\hat{a} = a$ and $k^a \in \mathbb{Z}_+^{A \cup \{\bar{c}\}}$ given by $k^a(a) = 1$, $k^a(\bar{c}) = L$, and $k^a(z) = 0$ otherwise. Indeed, before the step for a , set $x^a = \bar{p}(a)k^a$. The total amount subtracted from \bar{c} in all these steps is at most $L\bar{p}(A \setminus \{c\}) \leq L\eta < 1 - \eta \leq \bar{p}(\bar{c})$, so the construction is applicable at every step. Let \tilde{R} and $\tilde{p} \in F(\tilde{R})$ be the resulting profile and lottery on $A \cup \{\bar{c}\}$. The majority margin of c over \bar{c} is unchanged and therefore positive. Moreover, for each $a \in A \setminus \{c\}$,

$$M_{\tilde{R}}(c, a) = \frac{M_{\bar{R}}(c, a) + LM_{\bar{R}}(c, \bar{c})}{1 + L} \geq \frac{-1 + L\gamma/2}{1 + L} > 0.$$

Thus, c is a Condorcet winner for \tilde{R} . Finally, $\tilde{p}(c) = \bar{p}(c) = p(c) - x(c) < 1$, and $\tilde{p}(\bar{c}) > 0$ by the choice of η . Hence, the claim follows by setting $B = \{\bar{c}\}$. \square

Let $A \in \mathcal{F}(U)$, $c \in A$, $B \subseteq A \setminus \{c\}$, $\tilde{R} \in \mathcal{R}_A^*$, and $\tilde{p} \in F(\tilde{R})$ such that (i)–(iii) hold, which exist by Claim 1. Assume that $|\{a \in A \setminus (B \cup \{c\}) : \tilde{p}(a) > 0\}|$ is minimal among all possible choices of A , c , B , \tilde{R} , and \tilde{p} . Let $A^+ = \{a \in A \setminus (B \cup \{c\}) : \tilde{p}(a) > 0\}$. Assume for contradiction that A^+ is nonempty.

Since every alternative is a weak Condorcet winner for $\text{uni}(\mathcal{L}(A))$, Lemma 4 and convex-valuedness imply that $F(\text{uni}(\mathcal{L}(A))) = \Delta(A)$. Hence, by population-consistency, $\tilde{p} \in F(\lambda\tilde{R} + (1 - \lambda)\text{uni}(\mathcal{L}(A)))$ for each $\lambda \in [0, 1]$. We may thus assume that \tilde{R} is as close to $\text{uni}(\mathcal{L}(A))$ as we wish, and so we may assume that F is weakly Condorcet-consistent at \tilde{R} by Lemma 4. By the same argument, we will assume that F is weakly Condorcet-consistent at other profiles considered later in the proof without repeating the argument.

Let $\delta = \min\{\tilde{R}(c, a) : a \in A \setminus \{c\}\} - 1/2$, and note that $\delta > 0$ since c is a Condorcet winner. Let $a \in A^+$. Since F is weakly Condorcet-consistent at \tilde{R} and convex-valued, we have that $\lambda c + (1 - \lambda)\tilde{p} \in F(\tilde{R})$ for each $\lambda \in [0, 1]$. Hence, we may assume that $\tilde{p}(c)/\tilde{p}(a) = k \in \mathbb{N}$, where $k > 2\delta^{-1}$. Let $C \in \mathcal{F}(U)$ with $C \cap A = \{c\}$ and $|C| = k$, and let $\tilde{R}^+ \in \mathcal{R}_{A \cup C}^*$ be the profile resulting from \tilde{R} by replacing c with $\text{uni}(\mathcal{L}(C))$, so that $\tilde{R}^+|_A = \tilde{R}$, $\tilde{R}^+|_C = \text{uni}(\mathcal{L}(C))$, and C is a component in \tilde{R}^+ . By composition-consistency, $\tilde{p}^+ \in F(\tilde{R}^+)$, where $\tilde{p}^+ = \tilde{p} \times_c \text{uni}(C)$. Note that $\tilde{p}^+(a) = \tilde{p}^+(c')$ for each $c' \in C$ by the choice of k .

Let $\Pi \subseteq \Pi(A \cup C)$ contain all permutations π of $A \cup C$ with $\pi(C \cup \{a\}) = C \cup \{a\}$ and $\pi(z) = z$ for all $z \in A \setminus \{a, c\}$. Let

$$\hat{R}^+ = |\Pi|^{-1} \sum_{\pi \in \Pi} \pi(\tilde{R}^+).$$

By neutrality and population-consistency, $\tilde{p}^+ \in F(\hat{R}^+)$. Note that all alternatives in $C \cup \{a\}$ are clones in \hat{R}^+ , and each of them strictly dominates each alternative in $A \setminus \{a, c\}$ with a margin of at least $\delta/2$ by the choice of k . Let $\hat{R} = \hat{R}^+|_{A \setminus \{a\}}$ be the profile resulting from \hat{R}^+ by replacing the clones in $C \cup \{a\}$ by c , and let $\hat{p} \in \Delta(A \setminus \{a\})$ such that $\hat{p}(c) = \tilde{p}(c) + \tilde{p}(a)$ and $\hat{p}|_{A \setminus \{a, c\}} = \tilde{p}|_{A \setminus \{a, c\}}$. By composition-consistency, $\hat{p} \in F(\hat{R})$. Note that c is a Condorcet winner for \hat{R} , $\hat{p}(c) < 1$, and for all $b, b' \in B$, $\hat{p}(b) = \hat{p}(b') > 0$. Hence, $A \setminus \{a\}$, c , B , \hat{R} , and \hat{p} satisfy (i)–(iii) in Claim 1. This contradicts the minimality of $|A^+|$.

Thus, $\tilde{p}(a) = 0$ for all $a \in A \setminus (B \cup \{c\})$. Since c is a Condorcet winner and F satisfies weak Condorcet-consistency at \tilde{R} and convex-valuedness, we may assume that $\tilde{p}(c) \in (0, 1) \cap \mathbb{Q}$. Since $B \neq \emptyset$ and $\tilde{p}(b) = \tilde{p}(b')$ for all $b, b' \in B$, it follows that $\tilde{p} \in \Delta_{\mathbb{Q}}(A)$. The fact that $\tilde{p} \notin ML(\tilde{R}) = \{c\}$ completes the proof. \square

Lemma 6. *Let F be a PSCF that satisfies population-consistency and composition-consistency. Then, $F \subseteq ML$.*

Proof. Assume for contradiction that $F \not\subseteq ML$. By Lemma 5, there are $A \in \mathcal{F}(U)$, $R \in \mathcal{R}_A^*$, and $p \in F(R) \setminus ML(R)$ with $p \in \Delta_{\mathbb{Q}}(A)$. However, Lemma 3 together with Lemmas 6 to 13 of Brandl et al. (2016) rule out the existence of such a rational violation, a contradiction. Therefore, $F \subseteq ML$. \square

A.2. $ML \subseteq F$

We recall Lemma 14 of Brandl et al. (2016).

Lemma 7. *Let $M \in \mathbb{Q}^{n \times n}$ be a skew-symmetric matrix. Then, there are $R \in \mathcal{R}_{[n]}^*$ and $c \in \mathbb{Q}_{>0}$ such that $cM = M_R$. Furthermore, if there is $\pi \in \Pi([n])$ such that $M(i, j) = M(\pi(i), \pi(j))$ for all $i, j \in [n]$, then $R = \pi(R)$.*

Lemma 8. *Let F be a PSCF that satisfies population-consistency and composition-consistency. Then, $ML \subseteq F$.*

Proof. By Lemma 6, $F \subseteq ML$. Let $A \in \mathcal{F}(U)$, $R \in \mathcal{R}_A^*$, and $p \in ML(R)$. Because the vertices of the polytope $ML(R)$ are in $\Delta_{\mathbb{Q}}(A)$ and $F(R)$ is convex, we may assume that $p \in \Delta_{\mathbb{Q}}(A)$. By neutrality, we may assume without loss of generality that $A = [n]$ and $\text{supp}(p) = [k]$. Moreover, we may assume that k is odd: if k is even, introduce a clone of k and split the probability on k equally between k and its clone; the equal split ensures that $p \in \Delta_{\mathbb{Q}}(A)$.⁴ For simplicity, let $M = M_R$. It remains to show that $p \in F(R)$.

Consider the case that $k = 1$, i.e., there is $x \in A$ such that p is the degenerate lottery with probability 1 on x . ML only returns the degenerate lottery with probability 1 on x if x is a weak Condorcet winner. Therefore, x is a weak Condorcet winner for R . Let R' be a fractional preference profile for which x is a Condorcet winner, and for $\ell \in \mathbb{N}$, let $R_\ell = 1/\ell R' + (\ell-1)/\ell R$. Then x is a Condorcet winner for R_ℓ for each $\ell \in \mathbb{N}$. Therefore, by Lemma 6, $\emptyset \neq F(R_\ell) \subseteq ML(R_\ell) = \{p\}$ for each $\ell \in \mathbb{N}$, and continuity of F implies that $p \in F(R)$.

Now assume $k \geq 3$. By Lemma 7, there are $S \in \mathcal{R}_A^*$ and $c \in \mathbb{Q}_{>0}$ such that

$$M_S = c \begin{pmatrix} 0 & -\frac{1}{p_1 p_2} & 0 & \dots & 0 & \frac{1}{p_k p_1} & | & 1 & \dots & 1 \\ \frac{1}{p_1 p_2} & & & & & 0 & | & \vdots & \ddots & \vdots \\ 0 & & \ddots & & & \vdots & | & \vdots & \ddots & \vdots \\ \vdots & & & \ddots & & 0 & | & \vdots & \ddots & \vdots \\ 0 & & & & & -\frac{1}{p_{k-1} p_k} & | & 1 & \dots & 1 \\ -\frac{1}{p_k p_1} & 0 & \dots & 0 & \frac{1}{p_{k-1} p_k} & 0 & | & 1 & \dots & 1 \\ \hline -1 & & \dots & & & -1 & | & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots & | & \vdots & \ddots & \vdots \\ -1 & & \dots & & & -1 & | & 0 & \dots & 0 \end{pmatrix}$$

Intuitively, M_S defines a weighted cycle on $[k]$. Note that $(p^T M_S)_i = 0$ for all $i \in \text{supp}(p)$ and $(p^T M_S)_i > 0$ for all $i \in A \setminus \text{supp}(p)$, i.e., p is a quasi-strict maximin strategy in M_S in the sense of Harsanyi (1973). Since p is a maximin strategy in M_S , it follows that $p \in ML(S)$. For $\varepsilon \in [0, 1]$, we define $R^\varepsilon = (1 - \varepsilon)R + \varepsilon S$ and $M^\varepsilon = M_{R^\varepsilon}$. Population-consistency of ML implies that $p \in ML(R^\varepsilon)$ for all $\varepsilon \in [0, 1]$. Observe that p is a quasi-strict maximin strategy in M^ε for every $\varepsilon \in (0, 1]$. Hence, for every maximin strategy q in M^ε , it follows that $(q^T M^\varepsilon)_i = 0$ for every $i \in [k]$ and $q_i = 0$ for every $i \notin [k]$. It follows that

$$\det \left((M_S(i, j))_{i, j \in [k-1]} \right) = c^{k-1} \prod_{i=1}^{k-1} \frac{1}{p_i^2} \neq 0,$$

⁴The original proof by Brandl et al. (2016) treats the case of even k separately rather than using this simple reduction argument.

and hence, $(M_S(i, j))_{i, j \in [k]}$ has rank at least $k - 1$. In fact, $(M_S(i, j))_{i, j \in [k]}$ has rank $k - 1$, since skew-symmetric matrices of odd size cannot have full rank.⁵ Furthermore, $\det((M^\varepsilon(i, j))_{i, j \in [k-1]})$ is a nonzero polynomial in ε of degree at most $k - 1$ and hence, has at most $k - 1$ zeros. Thus, $(M^{1/\ell}(i, j))_{i, j \in [k]}$ has rank $k - 1$ for all but finitely many $\ell \in \mathbb{N}$. In particular, for all but finitely many $\ell \in \mathbb{N}$, $(q^T M^{1/\ell})_i = 0$ for all $i \in [k]$ and $q_i = 0$ for all $i \notin [k]$ imply that $q = p$. This implies that p is the unique maximin strategy in $M^{1/\ell}$ for all but finitely many $\ell \in \mathbb{N}$ and hence, $\emptyset \neq F(R^{1/\ell}) \subseteq ML(R^{1/\ell}) = \{p\}$ for all but finitely many $\ell \in \mathbb{N}$. Choose $\ell_0 \in \mathbb{N}$ such that $F(R^{1/\ell}) = \{p\}$ for all $\ell \geq \ell_0$, and let $R' = R^{1/\ell_0}$. For each $h \in \mathbb{N}$, $1/h R' + (1 - 1/h) R = R^{1/h\ell_0}$, so $F(1/h R' + (1 - 1/h) R) = \{p\}$, and continuity implies that $p \in F(R)$. \square

Proof of Theorem 2. Let f be a pSCF that satisfies population-consistency and composition-consistency. Let F be the PSCF induced by f . By Lemma 2, F satisfies population-consistency and composition-consistency. By Lemma 6 and Lemma 8, $F = ML$. Let $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}_A$ with $\sum_{\succ \in \mathcal{L}(A)} R(\succ) = n$, and let $R^* = 1/n R \in \mathcal{R}^*$. Then,

$$f(R) = F(R^*) = ML(R^*) = ML(R).$$

Therefore $f = ML$. \square

⁵A skew-symmetric matrix M of odd size cannot have full rank, since $\det(M) = \det(M^T) = \det(-M) = (-1)^n \det(M) = -\det(M)$ and, hence, $\det(M) = 0$.