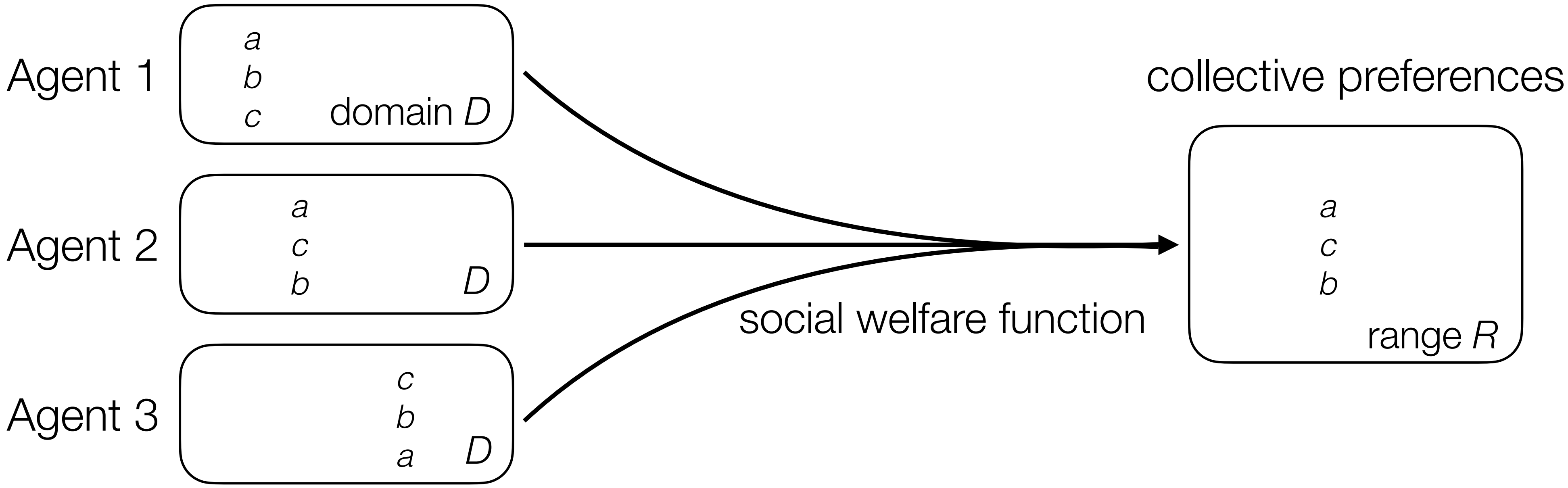
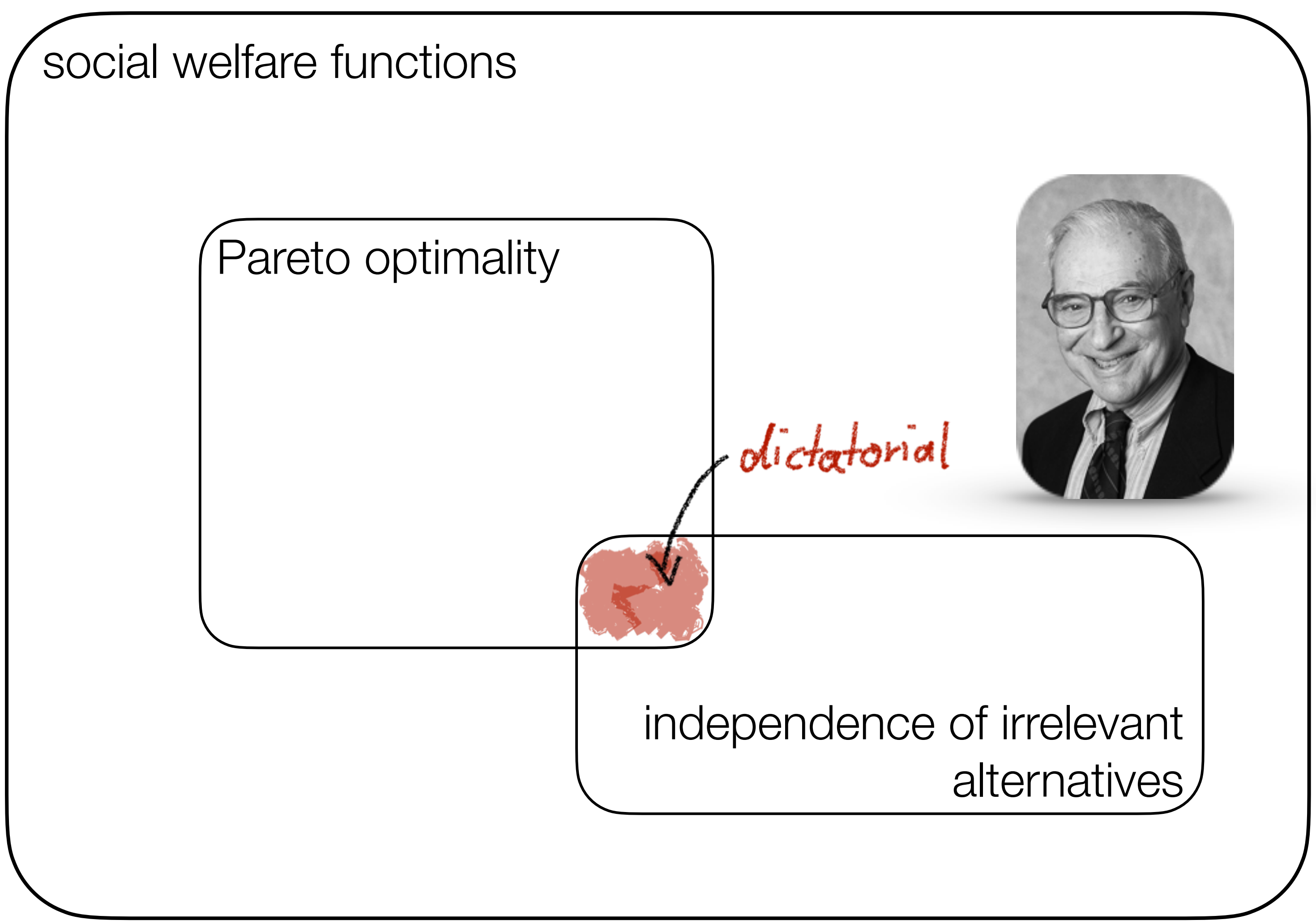
# Arrovian Aggregation of Convex Preferences

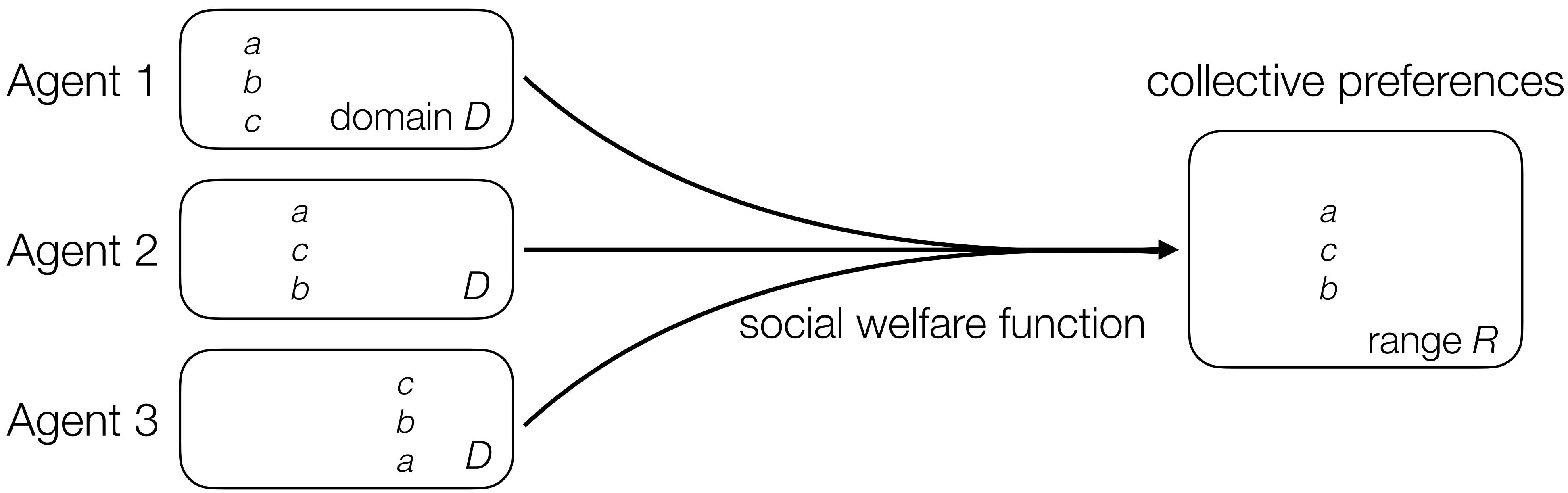
**Florian Brandl**
**Princeton University**

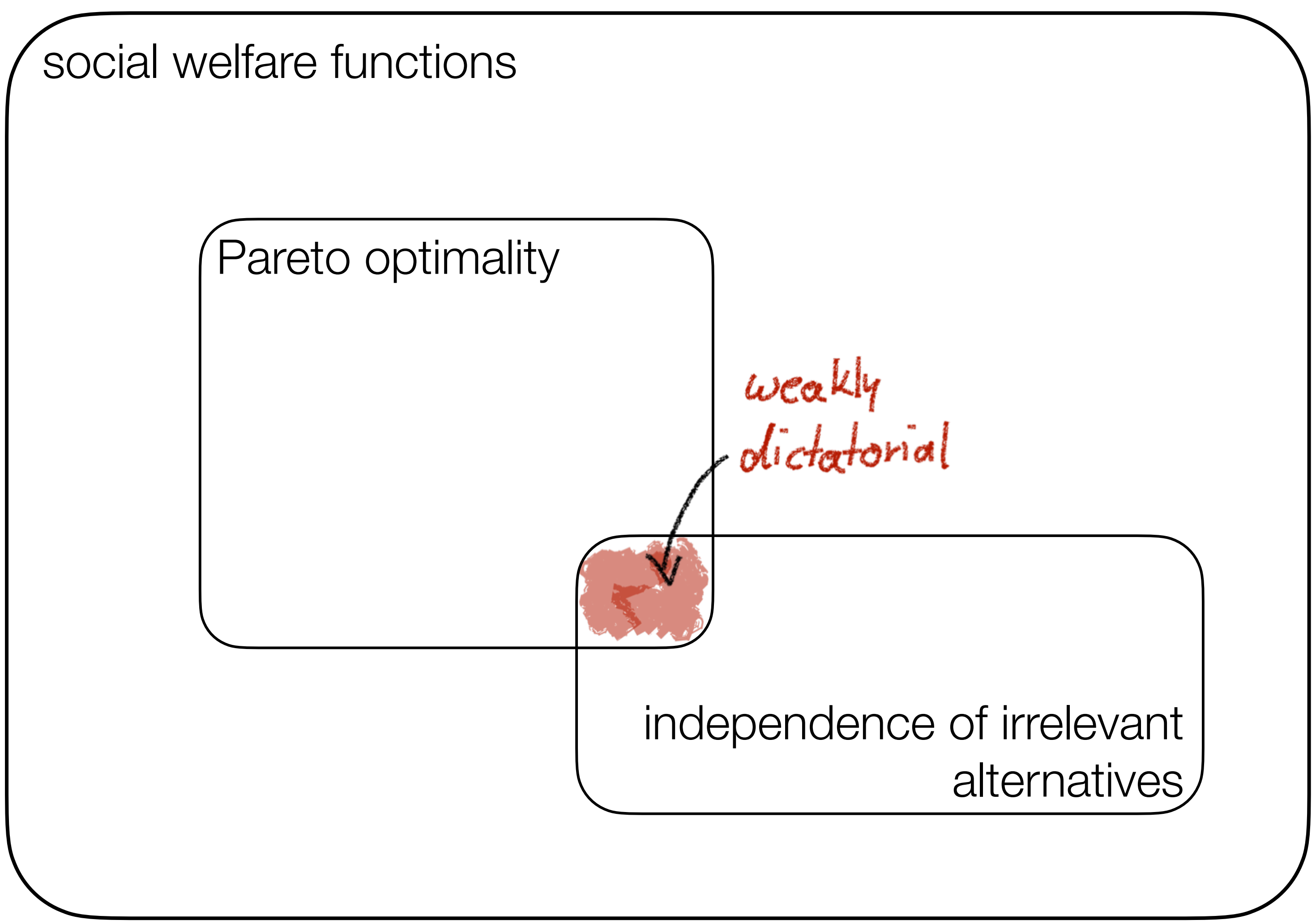
(joint with Felix Brandt)



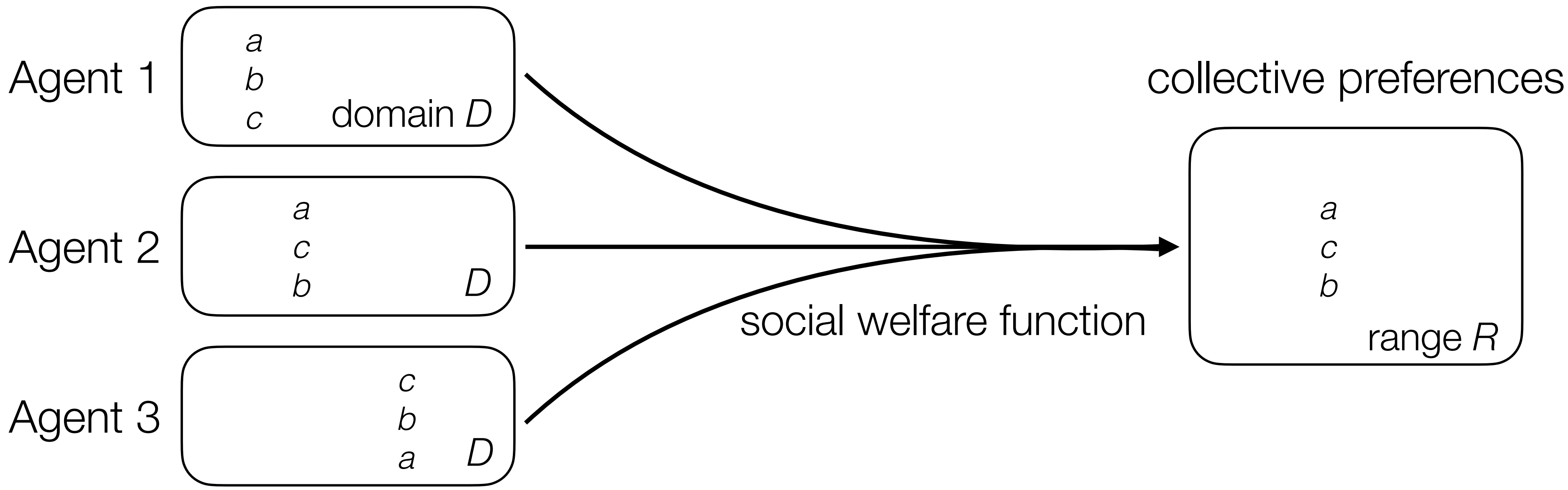



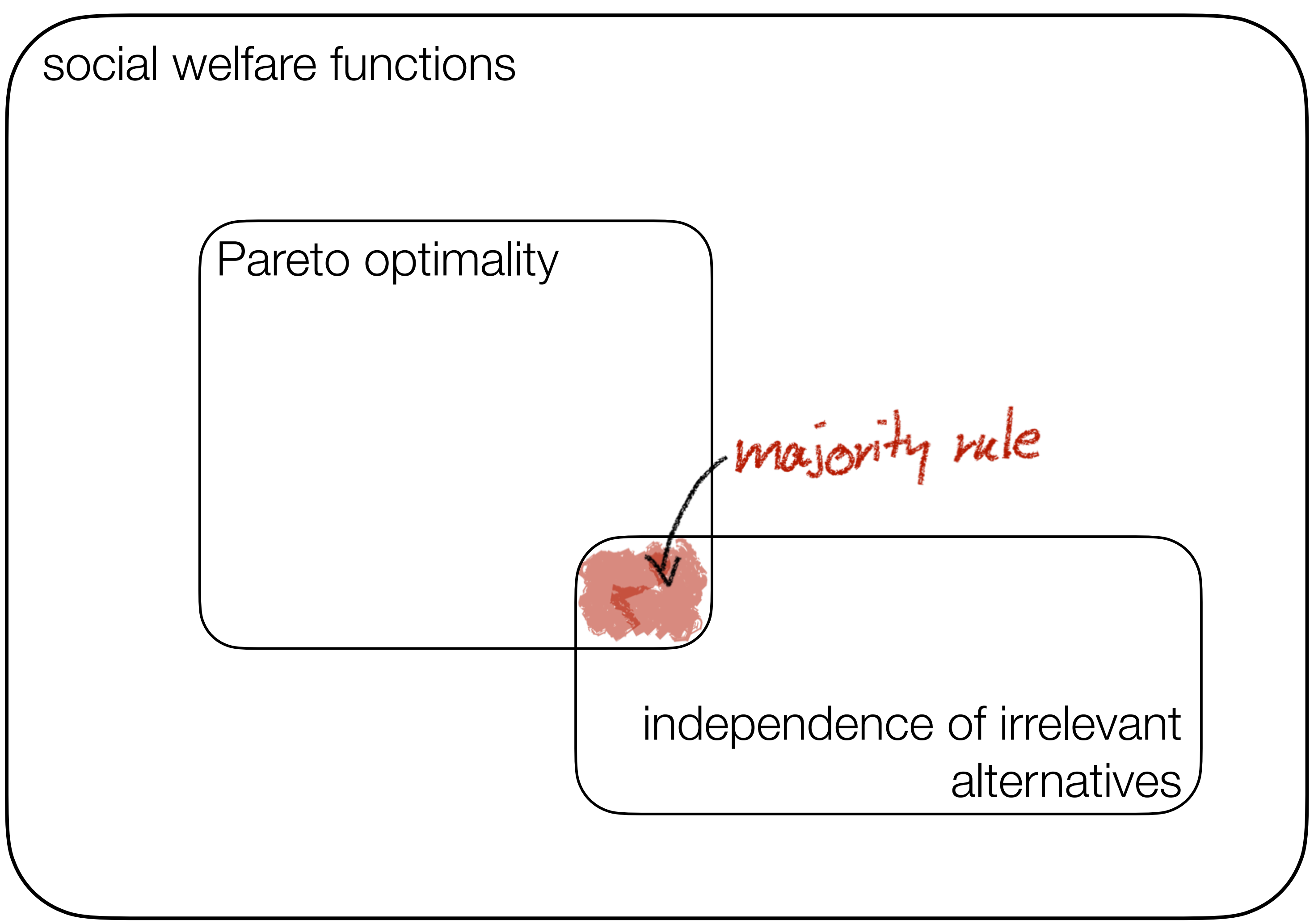

$D = R =$ transitive relations



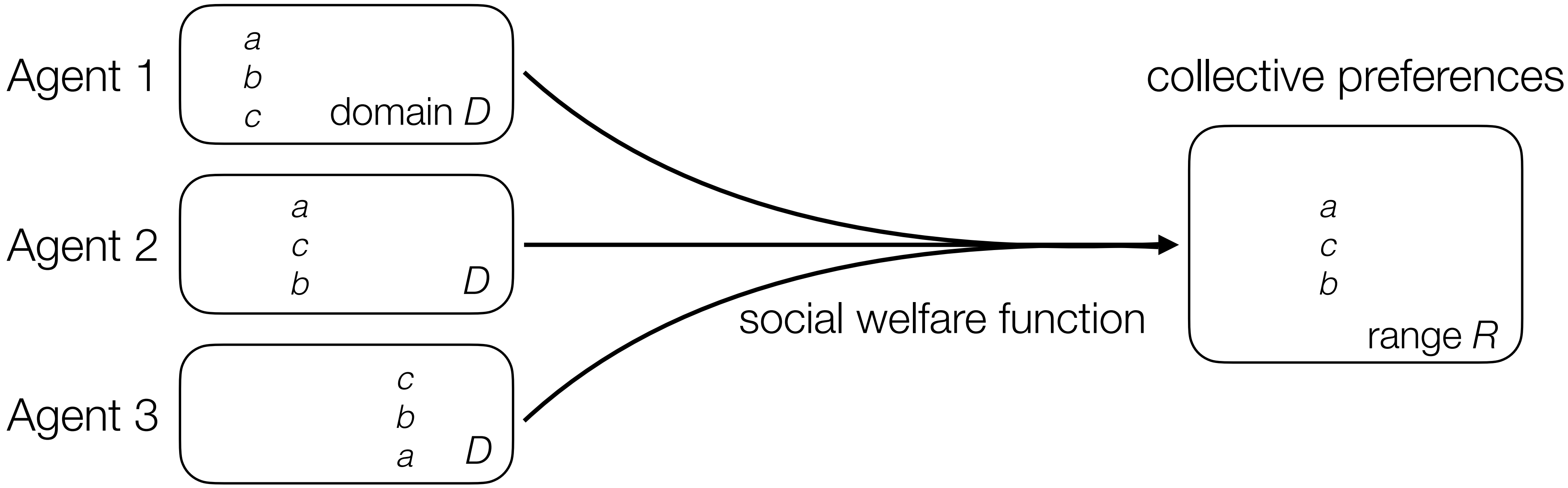



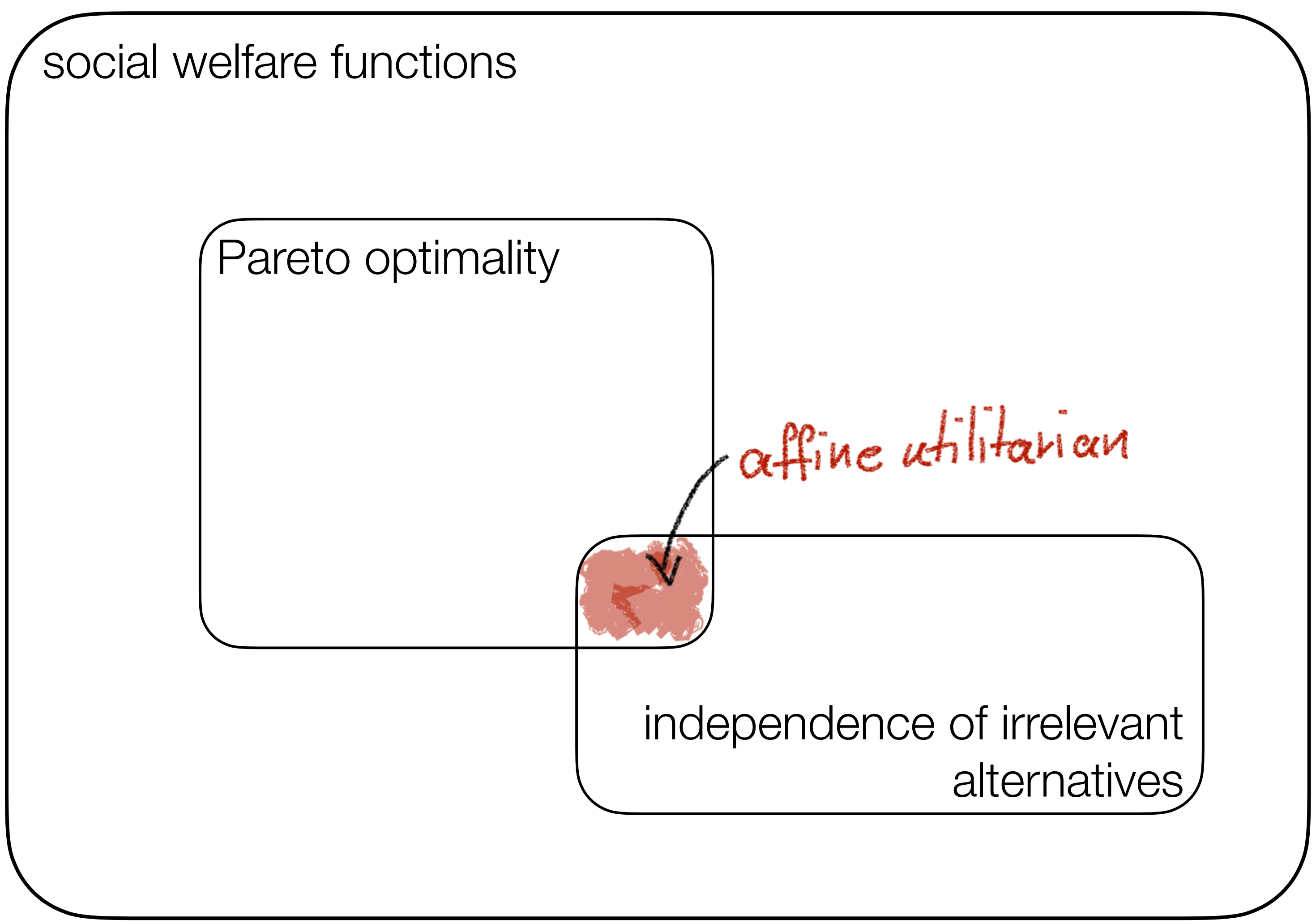

**extend *R*:** e.g., quasi-transitive relations




**restrict *D*:** e.g., single-peaked preferences



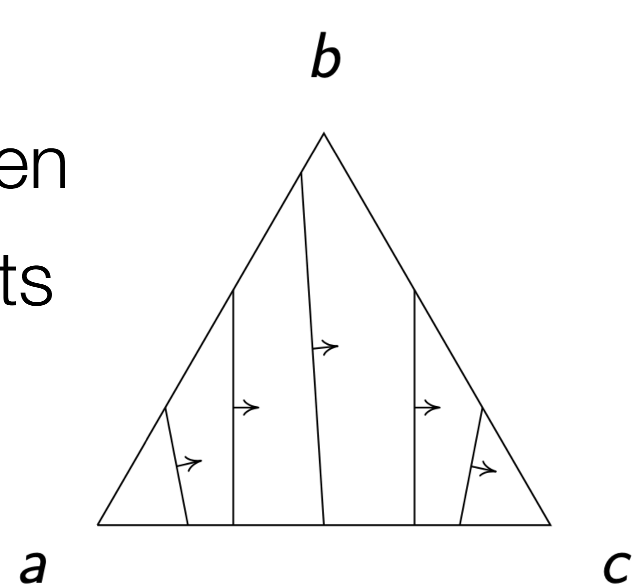

**this paper:** convex preferences

# Structured Outcome Sets

- Examples
  - Dividing a budget across projects
  - Assigning probabilities to deterministic options
  - Allocating time shares to tasks

$p = ½a + ½b$

- Alternatives $U = \{a,b,c,\dots\}$
  - Outcomes $\Delta U = \{p,q,r,\dots\}$ (convex combinations of alternatives)
- Preferences $\succ$ on $\Delta U$
  - Continuous if upper and lower contour sets are open
  - Convex if upper, lower, and indifference contour sets are convex



# What About Transitivity?

- **Sonnenschein (1971).** Continuous and convex relations admit maximal outcomes (in non-empty, compact, and convex sets)
  - Choosing maximal outcomes yields well-defined choice functions
- Sen's contraction (α) and expansion (γ) consistency: if an outcome is chosen …
  - … in some set, it is also chosen in any subset containing it
  - … in two sets, it is also chosen in (the convex hull of) their union
- **Proposition.** A continuous choice function
  - satisfies contraction and expansion consistency iff
  - it chooses maximal outcomes of a continuous and convex relation

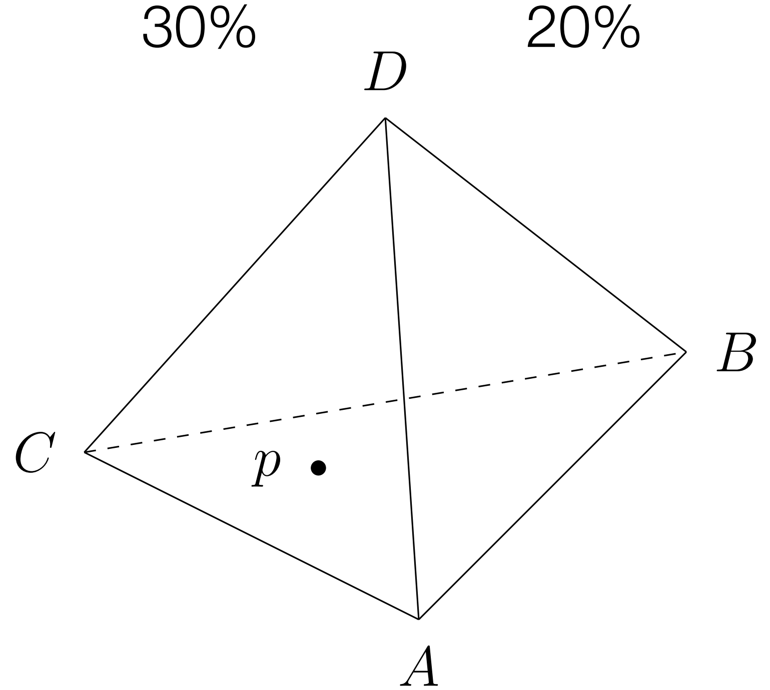
# Example: Budget Allocation

- 100 delegates vote on how to allocate a budget to public sectors
  - Each of 4 parties has made a proposal
  - Delegates have continuous and convex preferences over combinations of proposals

| | *Education* | *Transportation* | *Health* | *Military* |
|---|---|---|---|---|
| *A* | 40% | 30% | 20% | 10% |
| *B* | 30% | 10% | 40% | 20% |
| *C* | 20% | 30% | 30% | 20% |
| *D* | 10% | 30% | 20% | 40% |

contraction consistency

$p = \frac{1}{4}A + \frac{1}{4}B + \frac{1}{2}C$



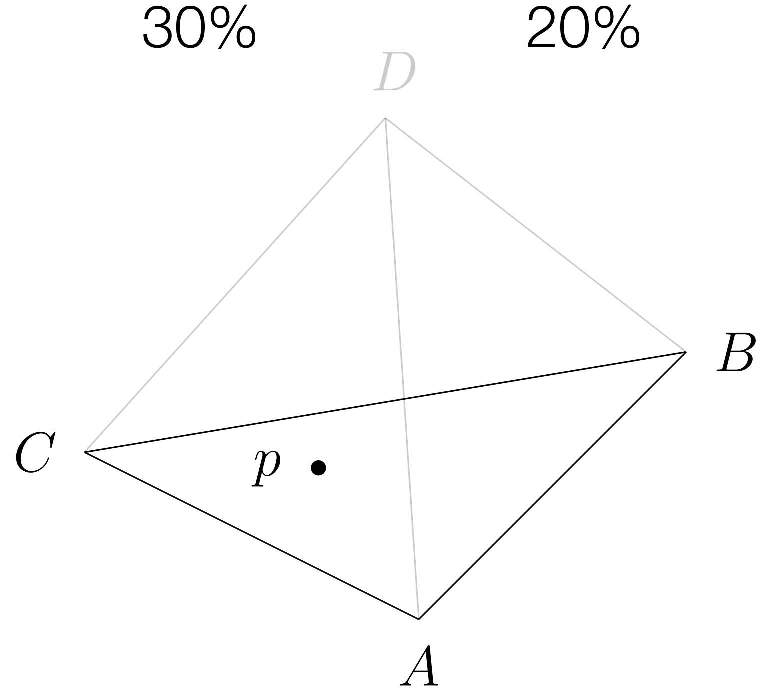
# Example: Budget Allocation

- 100 delegates vote on how to allocate a budget to public sectors
  - Each of 4 parties has made a proposal
  - Delegates have continuous and convex preferences over combinations of proposals

| | *Education* | *Transportation* | *Health* | *Military* |
|---|---|---|---|---|
| *A* | 40% | 30% | 20% | 10% |
| *B* | 30% | 10% | 40% | 20% |
| *C* | 20% | 30% | 30% | 20% |
| *D* | 10% | 30% | 20% | 40% |

contraction consistency

$p = \frac{1}{4}A + \frac{1}{4}B + \frac{1}{2}C$



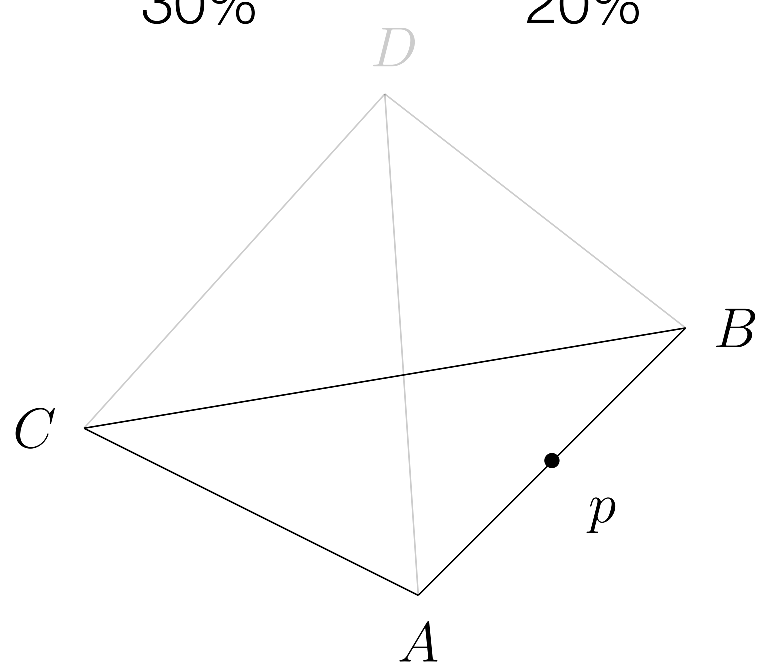
# Example: Budget Allocation

- 100 delegates vote on how to allocate a budget to public sectors
  - Each of 4 parties has made a proposal
  - Delegates have continuous and convex preferences over combinations of proposals

| | *Education* | *Transportation* | *Health* | *Military* |
|---|---|---|---|---|
| *A* | 40% | 30% | 20% | 10% |
| *B* | 30% | 10% | 40% | 20% |
| *C* | 20% | 30% | 30% | 20% |
| *D* | 10% | 30% | 20% | 40% |

expansion consistency

$p = ½A + ½B$



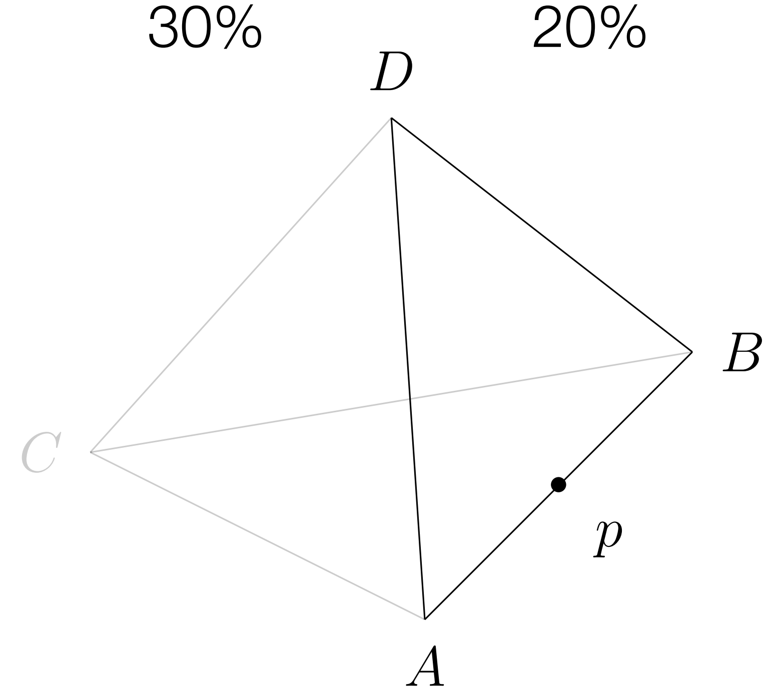
# Example: Budget Allocation

- 100 delegates vote on how to allocate a budget to public sectors
  - Each of 4 parties has made a proposal
  - Delegates have continuous and convex preferences over combinations of proposals

| | *Education* | *Transportation* | *Health* | *Military* |
|---|---|---|---|---|
| *A* | 40% | 30% | 20% | 10% |
| *B* | 30% | 10% | 40% | 20% |
| *C* | 20% | 30% | 30% | 20% |
| *D* | 10% | 30% | 20% | 40% |

expansion consistency

$p = ½A + ½B$



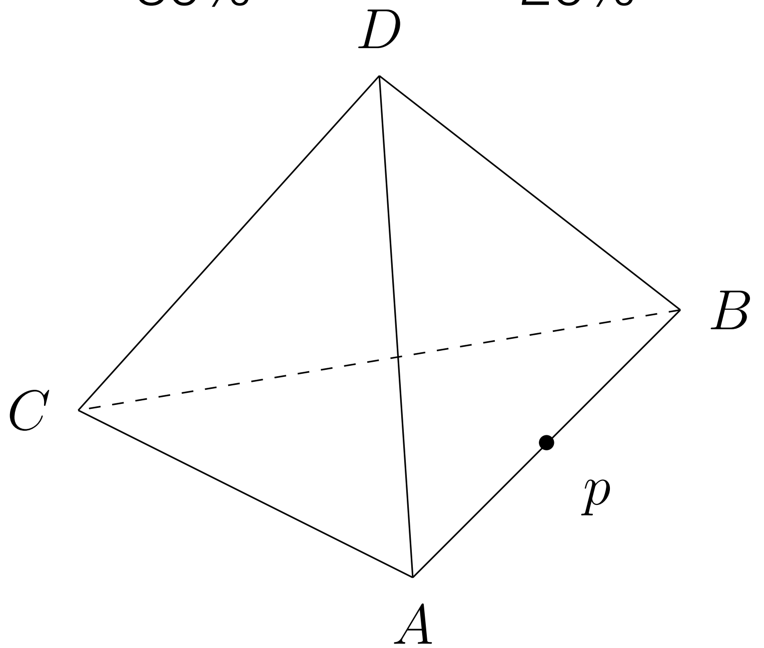
# Example: Budget Allocation

- 100 delegates vote on how to allocate a budget to public sectors
  - Each of 4 parties has made a proposal
  - Delegates have continuous and convex preferences over combinations of proposals

| | *Education* | *Transportation* | *Health* | *Military* |
|---|---|---|---|---|
| *A* | 40% | 30% | 20% | 10% |
| *B* | 30% | 10% | 40% | 20% |
| *C* | 20% | 30% | 30% | 20% |
| *D* | 10% | 30% | 20% | 40% |

expansion consistency

$p = ½A + ½B$



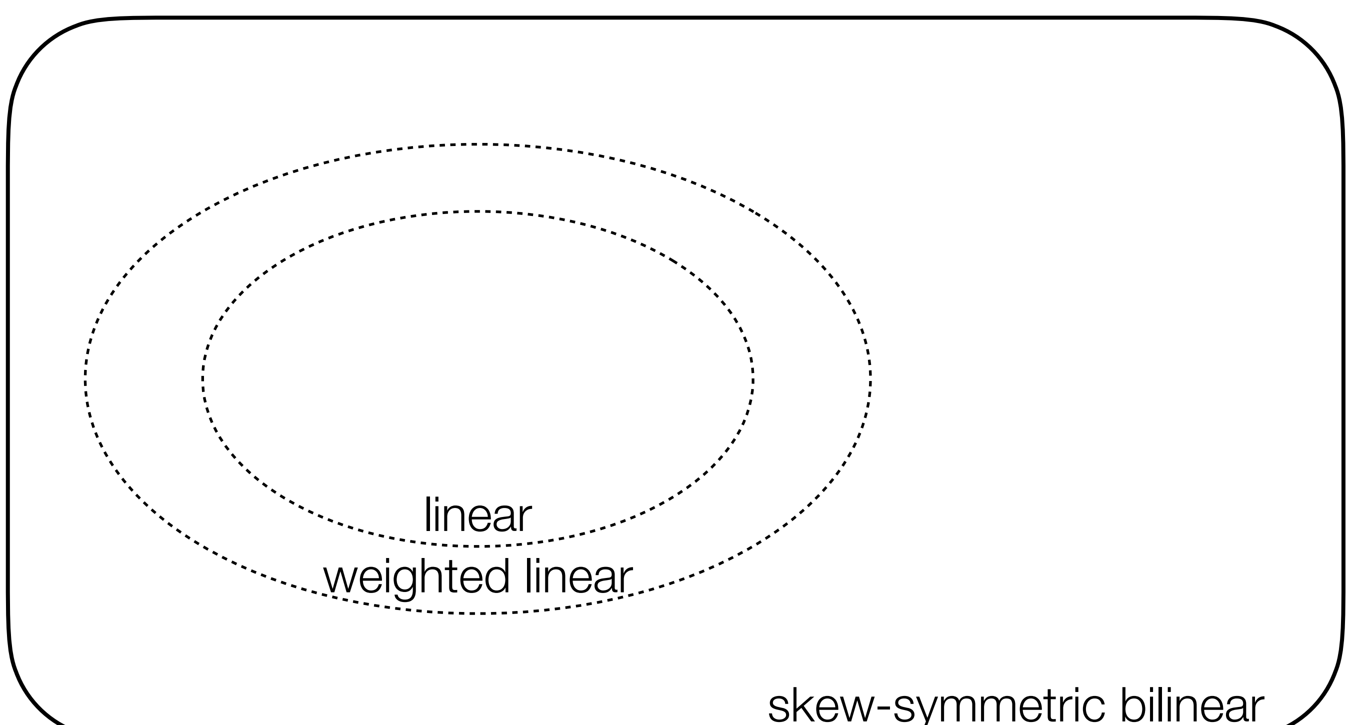
# Skew-Symmetric Bilinear Utilities

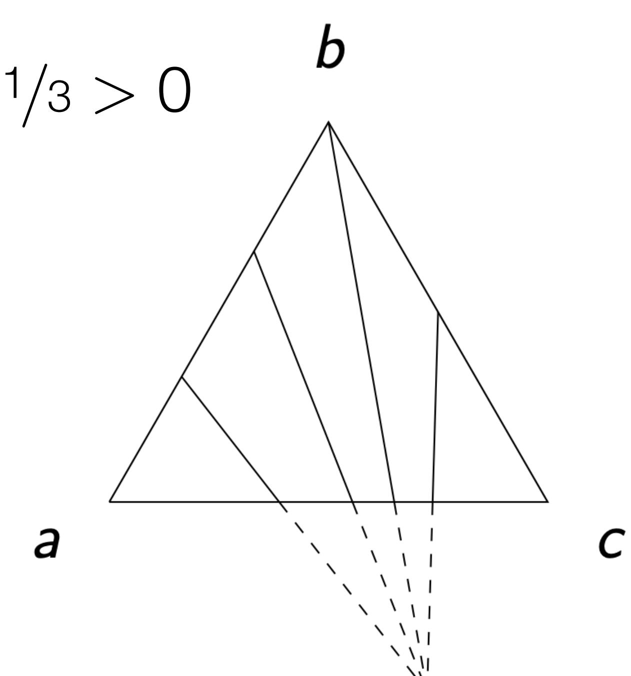
- **Proposition** (see also Fishburn, 1982).
  - $\succ$ is continuous, convex, and symmetric iff
  - there is a skew-symmetric matrix $\phi \in \mathbb{R}^{U \times U}$ such that for all $p,q \in \Delta U$,

$$p \succ q \iff p^T \phi q > 0$$

$$p^T \phi q = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 3 & 4 \\ -3 & 0 & 1 \\ -4 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 \\ 0 \\ 2/3 \end{pmatrix}$$

$$= 1/3 > 0$$



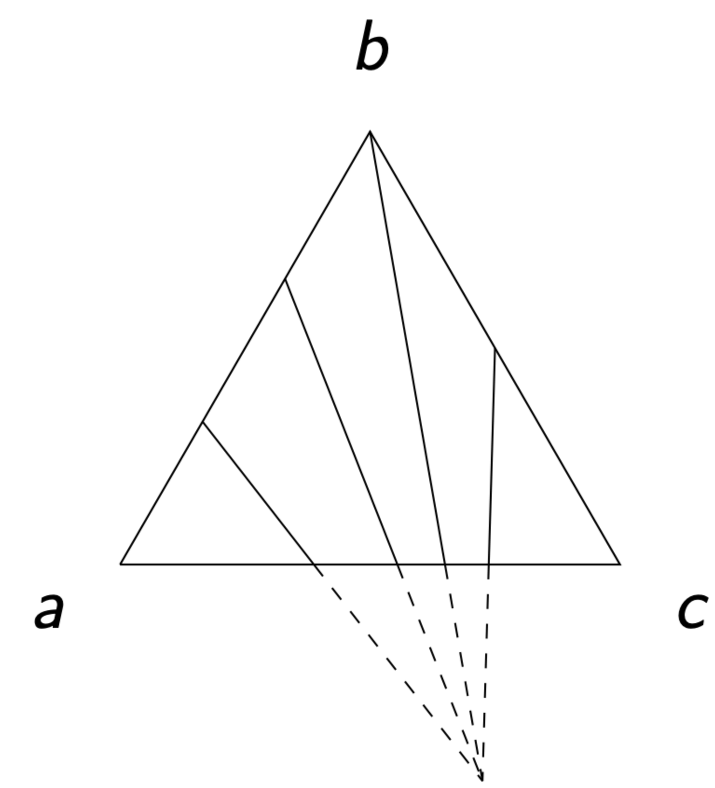



# Summary of the Model

- Alternatives $U = \{a, b, c, \ldots\}$
  - Outcomes $\Delta U = \{p, q, r, \ldots\}$

$p = ½a + ½b$

- Preference relations $R$ (continuous, convex, and symmetric)
  - Represented by skew-symmetric matrices
- Agents $N = \{1, \ldots, n\}$ with preferences $>_i$
  - Domain $D \subseteq R$ of agent preferences
  - Preference profiles $P \in D^N$
- Social welfare function $f$: $D^N \rightarrow R$



An Arrovian social welfare function $f$ satisfies

- Pareto optimality: for all $p,q \in \Delta U$ and $P \in D^N$ with $\succsim = f(P)$,

$$(\forall\, i \in N)\; p \succsim_i q \;\Rightarrow\; p \succsim q$$

If $p \succ_i q$ for some $i \in N$, then $p \succ q$

- independence of irrelevant alternatives: for all $A \subseteq U$ and $P, P' \in D^N$,

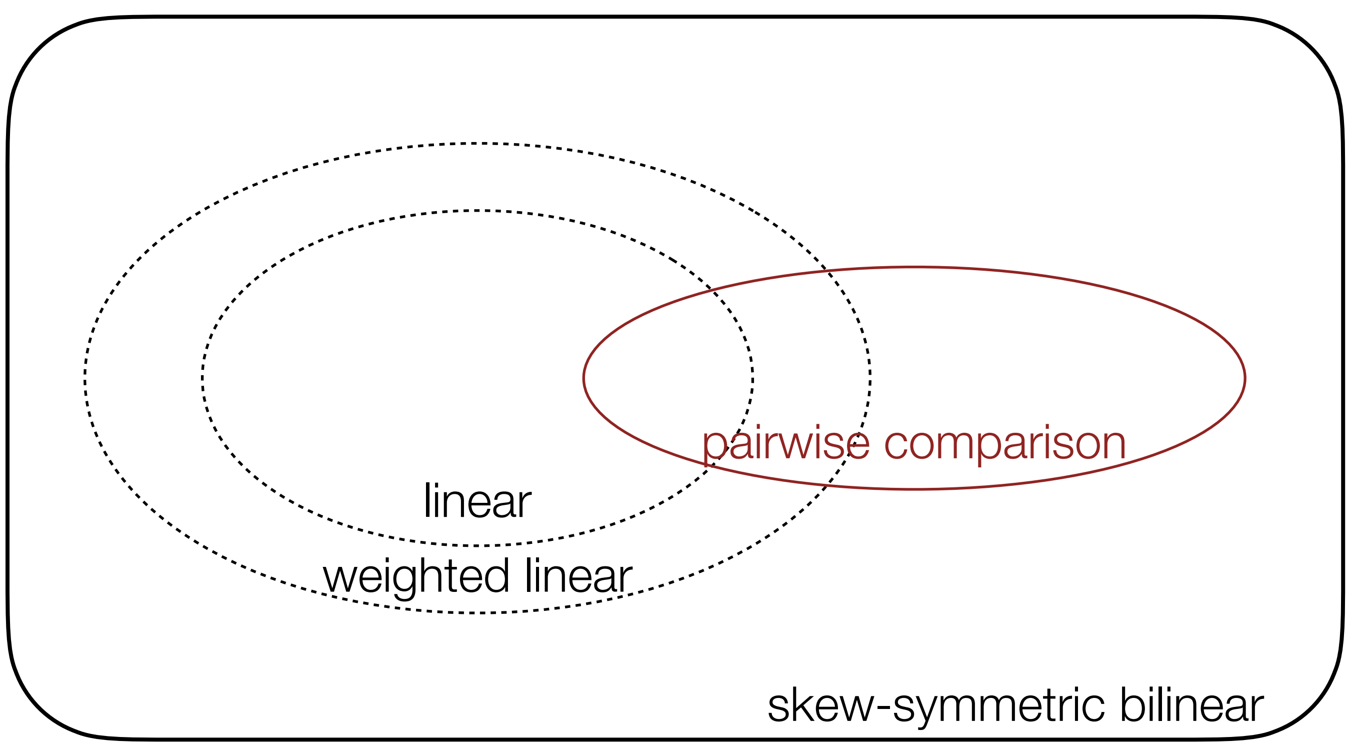
$$P|_{\Delta A} = P'|_{\Delta A} \;\Rightarrow\; f(P)|_{\Delta A} = f(P')|_{\Delta A}$$

# Arrovian Domains

- Pairwise comparison $D^{PC} = \{\succ \in R \colon \succ \cong \phi \text{ with } \phi \in \{-1,0,1\}^{|U|\times|U|}\}$
  - $p \succ q$ if $p$ is more likely to return a better alternative than $q$
  - Observed in experiments (Blavatskyy, 2006; Butler & Pogrebna, 2018)

**Theorem.** $|U| \geq 4$, $D \subseteq R$ rich, $f\colon D^N \to R$ anonymous and Arrovian

$$\Rightarrow D \subseteq D^{PC}$$




$$p^T \phi q = \begin{matrix} & \begin{matrix} 1/3 & 0 & 2/3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \end{matrix}$$

$$= 1/3 > 0$$

# Arrovian Social Welfare Functions

**Theorem.** $|U| \geq 5$, $D \subseteq D^{PC}$ rich, $f$: $D^N \rightarrow R$ Arrovian

$$\Rightarrow f(P) \cong \sum w_i \phi_i$$

for some $w_i > 0$, where $\phi_i \cong >_i$

**Corollary.** $|U| \geq 5$, $D \subseteq R$ rich, $f$: $D^N \rightarrow R$ anonymous and Arrovian

$$\Rightarrow D \subseteq D^{PC} \quad \text{and} \quad f(P) \cong \sum \phi_i$$

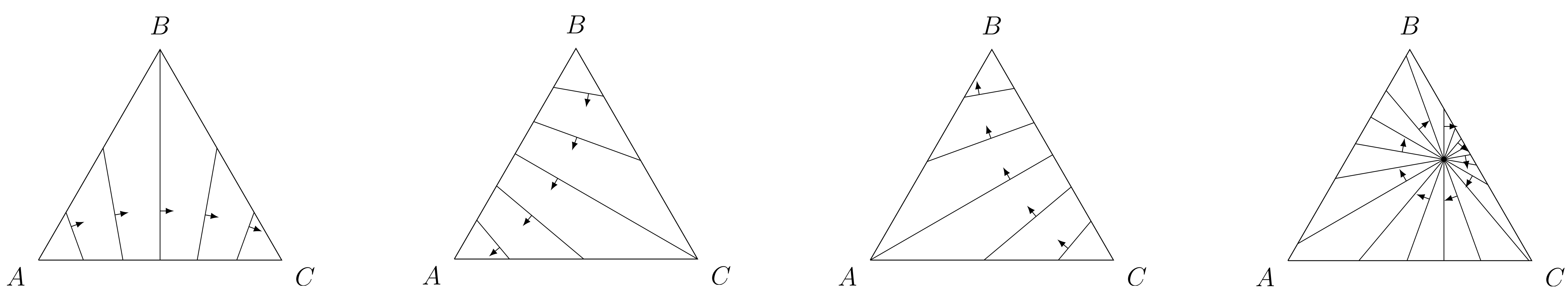
Society prefers $p$ to $q$ if a random agent more likely than not prefers the alternative chosen by $p$ to the one chosen by $q$

# Example: Budget Allocation

| | *Education* | *Transportation* | *Health* | *Military* |
|---|---|---|---|---|
| *A* | 40% | 30% | 20% | 10% |
| *B* | 30% | 10% | 40% | 20% |
| *C* | 20% | 30% | 30% | 20% |
| *D* | 10% | 30% | 20% | 40% |

| 25 | 30 | 45 |
|---|---|---|
| *A* | *B* | *C* |
| *B* | *C* | *A* |
| *C* | *A* | *B* |

$$25\begin{array}{c}\begin{matrix} A & B & C \end{matrix}\\ \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}\end{array}\begin{matrix} \\ A \\ B \\ C \end{matrix} + 30\begin{array}{c}\begin{matrix} A & B & C \end{matrix}\\ \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\end{array}\begin{matrix} \\ A \\ B \\ C \end{matrix} + 45\begin{array}{c}\begin{matrix} A & B & C \end{matrix}\\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}\end{array}\begin{matrix} \\ A \\ B \\ C \end{matrix} = \begin{array}{c}\begin{matrix} A & B & C \end{matrix}\\ \begin{pmatrix} 0 & 40 & -50 \\ -40 & 0 & 10 \\ 50 & -10 & 0 \end{pmatrix}\end{array}\begin{matrix} \\ A \\ B \\ C \end{matrix}$$

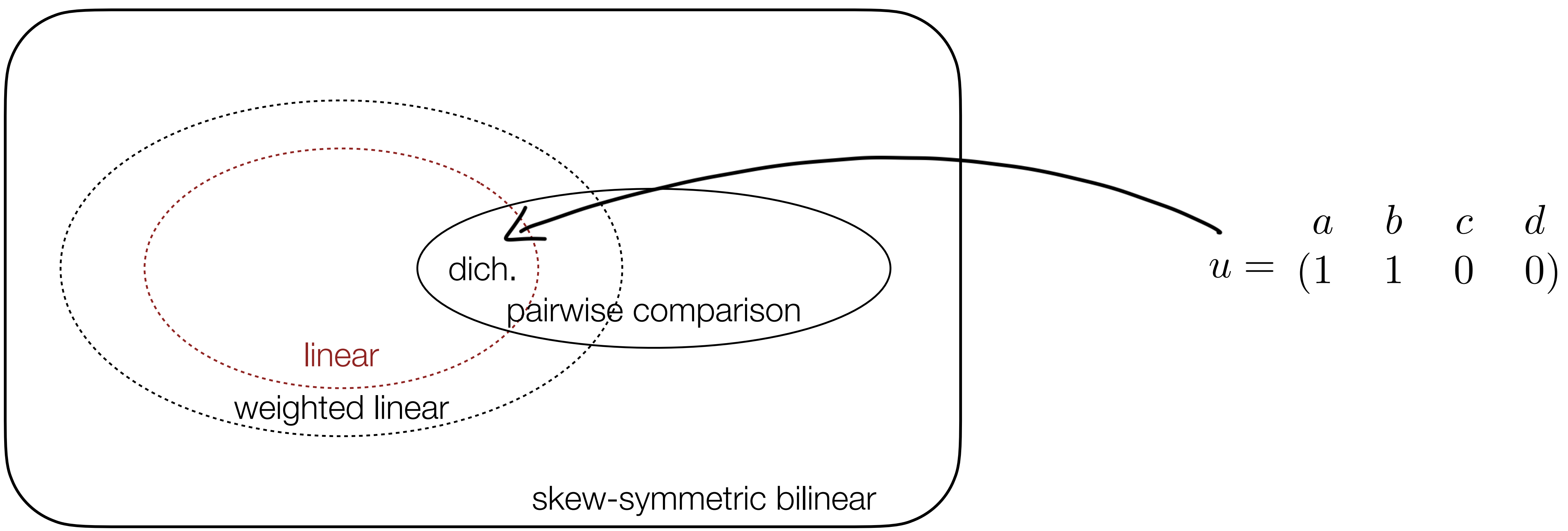



maximal outcome 0.1*A* + 0.5*B* + 0.4*C*

# Aggregation of vNM Preferences

**Theorem.** $|U| \geq 4$, $D \subseteq R^{vNM}$ rich, $f$: $D^N \rightarrow R^{vNM}$ anonymous and Arrovian

$$\Rightarrow D \subseteq D^{dich} \quad \text{and} \quad f(P) \cong \sum u_i$$




Society prefers $p$ to $q$ if a random agent is more likely to approve the alternative chosen by $p$ than that chosen by $q$

# Convex Preferences Permit Arrovian Aggregation

We have characterized

- choice functions that give rise to continuous and convex preferences
- the domain of preferences that allows for anonymous Arrovian aggregation
- the class of Arrovian social welfare functions