Justifying Optimal Play via Consistency

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(joint work with Felix Brandt)

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Introduction

In two-player zero-sum games, von Neumann's minimax theorem gives strong reasons to play maximin strategies.

- Pairs of maximin strategies correspond to Nash equilibria.
- Maximin strategies do not require coordination among players (unlike Nash equilibria in general normal form games).
- Still, providing normative foundations for maximin play turned out to be surprisingly difficult.

Goal: Characterize maximin strategies via coherent behavior in varying games.

> A rational and consistent consequentialist who ascribes the same properties to his opponent must play maximin strategies.

Outline

The Model

- U an infinite universal set of actions $\qquad U = \{a, b, c, \dots\}$
  - $\mathcal{F}(U)$ the set of *finite* subsets of U $\qquad A = \{a, b, c\} \in \mathcal{F}(U)$
  - $\Delta(A)$ the set of *rational-valued* strategies over $A \in \mathcal{F}(U)$

    $p = (1/2, 1/2, 0) \in \Delta(A)$
- $M \in \mathbb{Q}^{A \times B}$ a zero-sum game with action sets $A, B \in \mathcal{F}(U)$

  $$M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$
- A solution concept maps a game $M \in \mathbb{Q}^{A \times B}$ to a set of optimal strategies $f(M) \subseteq \Delta(A)$ for the row player
  - $\mathit{maximin}(M) = \arg\max_{p \in \Delta(A)} \min_{q \in \Delta(B)} p^t M q$

    $\mathit{maximin}(M) = \{(2/3, 1/3)\}$

Consequentialism

Players do not distinguish between payoff-equivalent actions.

For $\hat{A} \subseteq A \in \mathcal{F}(U)$, $\hat{B} \subseteq B \in \mathcal{F}(U)$, $M \in \mathbb{Q}^{A \times B}$, and $\hat{M} \in \mathbb{Q}^{\hat{A} \times \hat{B}}$, $\hat{M}$ is a reduced form of M if there exist surjective functions $\alpha\colon A \to \hat{A}$ and $\beta\colon B \to \hat{B}$ such that, for all $(a, b) \in A \times B$, $M_{ab} = \hat{M}_{\alpha(a)\beta(b)}$.

$$f(M) = \bigcup_{\hat{p} \in f(\hat{M})} \{p \in \Delta(A)\colon \sum_{a \in \alpha^{-1}(\hat{a})} p(a) = \hat{p}(\hat{a}) \text{ for all } \hat{a} \in \hat{A}\}$$

Consequentialism

For $\hat{A} \subseteq A \in \mathcal{F}(U)$, $\hat{B} \subseteq B \in \mathcal{F}(U)$, $M \in \mathbb{Q}^{A \times B}$, and $\hat{M} \in \mathbb{Q}^{\hat{A} \times \hat{B}}$, $\hat{M}$ is a reduced form of M if there exist surjective functions $\alpha \colon A \to \hat{A}$ and $\beta \colon B \to \hat{B}$ such that, for all $(a, b) \in A \times B$, $M_{ab} = \hat{M}_{\alpha(a)\beta(b)}$.

$$f(M) = \bigcup_{\hat{p} \in f(\hat{M})} \{p \in \Delta(A) \colon \sum_{a \in \alpha^{-1}(\hat{a})} p(a) = \hat{p}(\hat{a}) \text{ for all } \hat{a} \in \hat{A}\}$$

Example:

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \qquad \hat{M} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$f(M) = \{(2/3, \lambda, 1/3 - \lambda) \colon \lambda \in [0, 1/3]\} \quad f(\hat{M}) = \{(2/3, 1/3)\}$$

Consistency

If a strategy is optimal in two strategically similar games it is also optimal if there is uncertainty which of them is played.

For all $A, B \in \mathcal{F}(U)$, $\lambda \in [0,1] \cap \mathbb{Q}$, and $\hat{M}, \bar{M} \in \mathbb{Q}^{A \times B}$, if $f(\hat{M}) \cap f(\bar{M}) \neq \emptyset$ and $f(-\hat{M}^t) \cap f(-\bar{M}^t) \neq \emptyset$, then

$$f(\hat{M}) \cap f(\bar{M}) \subseteq f(\lambda \hat{M} + (1-\lambda)\bar{M}).$$

Consistency

For all $A, B \in \mathcal{F}(U)$, $\lambda \in [0,1] \cap \mathbb{Q}$, and $\hat{M}, \bar{M} \in \mathbb{Q}^{A \times B}$, if $f(\hat{M}) \cap f(\bar{M}) \neq \emptyset$ and $f(-\hat{M}^t) \cap f(-\bar{M}^t) \neq \emptyset$, then

$$f(\hat{M}) \cap f(\bar{M}) \subseteq f(\lambda \hat{M} + (1-\lambda)\bar{M}).$$

Example:

$$\hat{M} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \quad \bar{M} = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 0 & 4 \\ 4 & 0 & 0 \end{pmatrix} \quad {}^1\!/_2 \hat{M} + {}^1\!/_2 \bar{M} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 2 \end{pmatrix}$$

$$f(\hat{M}) = f(\bar{M}) = \{({}^2\!/_5, {}^2\!/_5, {}^1\!/_5)\}$$

$$f(-\hat{M}^t) = f(-\bar{M}^t) = \{({}^2\!/_5, {}^1\!/_5, {}^2\!/_5)\}$$

$$({}^2\!/_5, {}^2\!/_5, {}^1\!/_5) \in f({}^1\!/_2\, \hat{M} + {}^1\!/_2\, \bar{M})$$

Rationality

Strictly dominated actions are never played.

For $A, B \in \mathcal{F}(U)$ and $M \in \mathbb{Q}^{A \times B}$, $a \in dom(M)$ if there is $a' \in A$ with $M_{ab} < M_{a'b}$ for all $b \in B$.

$$f(M) \subseteq \{p \in \Delta(A) \colon p(dom(M)) = 0\}$$

Example:

$$M = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$f(M) \subseteq \{(\lambda, 1-\lambda, 0) \colon \lambda \in [0,1]\}$$

Result

If a solution concept f satisfies consequentialism, consistency, and rationality, then $f \subseteq \mathit{maximin}$.

Proof idea:

- If one of the players does not play a maximin strategy, their strategies do not constitute a Nash equilibrium.
- Use consequentialism and consistency to construct a game where the player who has a profitable deviation plays a dominated action.
- Apply rationality to get a contradiction.

Proof Idea

$$
\begin{array}{c} \\ 2/3 \\ 1/3 \\ \\ \end{array}
\begin{array}{c} \begin{array}{cc} 2/3 & 1/3 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}
\xrightarrow{\text{conseq.}}
\begin{array}{c} \\ \\ 1/3 \\ 1/3 \\ 1/3 \\ \\ \end{array}
\begin{array}{c} \begin{array}{cc} 2/3 & 1/3 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}
\xrightarrow{\text{consist.}}
\begin{array}{c} \\ \\ 1/3 \\ 1/3 \\ 1/3 \\ \\ \end{array}
\begin{array}{c} \begin{array}{cc} 2/3 & 1/3 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 2/3 & 1/3 \\ 2/3 & 1/3 \\ 2/3 & 1/3 \\ -1 & 0 \end{pmatrix} \end{array}
$$

$$
\begin{array}{c} \\ 1/3 \\ 1/3 \\ 1/3 \\ \\ \end{array}
\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}
$$

$$
\begin{array}{c} \\ 1/3 \\ 1/3 \\ 1/3 \\ \\ \end{array}
\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}
$$

Proof Idea

$$
\begin{array}{cc} & \begin{array}{cc} 2/3 & 1/3 \end{array} \\ \begin{array}{c} 2/3 \\ \color{orange}{1/3} \\ \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}
\xrightarrow{\text{conseq.}}
\begin{array}{cc} & \begin{array}{cc} 2/3 & 1/3 \end{array} \\ \begin{array}{c} \\ 1/3 \\ 1/3 \\ \color{orange}{1/3} \\ \end{array} & \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}
\xrightarrow{\text{consist.}}
\begin{array}{cc} & \begin{array}{cc} 2/3 & 1/3 \end{array} \\ \begin{array}{c} \\ \color{orange}{1/3} \\ \color{orange}{1/3} \\ \color{orange}{1/3} \\ \end{array} & \begin{pmatrix} 1 & 0 \\ 2/3 & 1/3 \\ 2/3 & 1/3 \\ 2/3 & 1/3 \\ -1 & 0 \end{pmatrix} \end{array}
\xrightarrow{\text{conseq.}}
$$

$$
\begin{array}{cc} & \begin{array}{cc} 2/3 & 1/3 \end{array} \\ \begin{array}{c} \\ \color{orange}{1} \\ \end{array} & \begin{pmatrix} 1 & 0 \\ 2/3 & 1/3 \\ -1 & 0 \end{pmatrix} \end{array}
\xrightarrow{\text{conseq.}}
\begin{array}{cc} & \begin{array}{cccc} & 1/3 & 1/3 & 1/3 \end{array} \\ \begin{array}{c} \\ \color{orange}{1} \\ \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2/3 & 2/3 & 2/3 & 1/3 \\ -1 & -1 & -1 & 0 \end{pmatrix} \end{array}
\xrightarrow[\text{consist.}]{\text{conseq.}}
\begin{array}{cc} & \begin{array}{cc} & 1 \end{array} \\ \begin{array}{c} \\ \color{orange}{1} \\ \end{array} & \begin{pmatrix} 1 & 2/3 \\ 2/3 & 5/9 \\ -1 & -2/3 \end{pmatrix} \end{array}
$$

The row player plays a dominated action.

Related Work

- Epistemic approaches: Bayesian belief hierarchies among players, which capture their knowledge/beliefs about other players (e.g., Aumann and Brandenburger, 1995; Aumann and Drèze, 2008).
- Characterizations of the value: Typically not motivated on normative grounds; do not carry strategic content (e.g., Vilkas, 1963; Tijs, 1981; Norde and Voorneveld, 2004).
- Preferences over game forms: Imply existence of an underlying utility function so that game forms are ranked according to the value of the resulting games (Hart et al., 1994).

Extension to Bimatrix Games

A bimatrix game $(M, N) \in (\mathbb{Q}^{A\times B})^2$ specifies a payoff matrix for each player.

- (M, N) is zero-sum if $M + N = 0$.
- Nash equilibria are not interchangeable for bimatrix games: if $(p, q), (p', q')$ are equilibria, (p, q') may not be an equilibrium.
- Consequentialism, consistency, and rationality carry over to solution concepts mapping bimatrix games to sets of recommended strategies for the row player.

If a solution concept satisfies our axioms, any strategy profile consisting of one recommended strategy for each player constitutes a Nash equilibrium.
This characterizes solution concepts returning interchangeable sets of Nash equilibria (and Nash equilibrium in solvable games (Nash, 1951)).

Discussion

- Consequentialism, consistency, and rationality are all required for the characterization of *maximin*.
- The characterization remains valid in the domain of symmetric zero-sum games.
- *maximin* violates strong consistency defined as follows: $f(\hat{M}) \cap f(\bar{M}) \neq \emptyset$ implies $f(\hat{M}) \cap f(\bar{M}) \subseteq f(\lambda\hat{M} + (1-\lambda)\bar{M})$. (Consistency additionally requires $f(-\hat{M}^t) \cap f(-\bar{M}^t) \neq \emptyset$.)
- Extensions to games with real-valued payoffs are feasible under additional assumptions about the set of actions or the solution concept, but not straightforward.