

Belief-Averaged Relative Utilitarianism

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We study preference aggregation under uncertainty when individual and collective preferences are based on subjective expected utility. A natural procedure for determining the collective preferences of a group then is to average its members' beliefs and add up their $(0, 1)$ -normalized utility functions. This procedure extends the well-known relative utilitarianism to decision making under uncertainty. We show that it is the only aggregation function that gives tie-breaking rights to agents who join a group and satisfies an independence condition in the spirit of Arrow's independence of irrelevant alternatives as well as four undiscriminating axioms.

A group of policymakers seeks to decide between action plans whose consequences depend on unknown facts about the world. Their preferences over the alternatives hinge on their beliefs about the world, such as the probability that an economic intervention is efficacious, and their tastes for consequences like enhanced labor rights or higher GDP. How should they arrive at a group preference?

To put this question on formal grounds, we assume that each agent is rational in the sense that her preferences meet Savage's (1954) criteria and we shall hold groups to the same standard. The beauty of Savage's assumptions is that they allow us to express preferences in an intuitive way: agents assign probabilities to possible states of the world—expressing their belief—and utilities to consequences—capturing tastes, so that more preferred acts have higher expected utility. To tackle the aggregation problem, we consider aggregation functions, which map each possible configuration of preferences for the members of a group to a preference of the group. Our approach is thus multi-profile in that it contemplates hypothetical configurations that could arise rather than a single observed one.

A seemingly sensible way to arrive at the preferences of a group—or, equivalently, its belief and utility function—is by averaging its members' beliefs and adding up their utility functions, each normalized so that the minimal and maximal values agree across agents. We call this aggregation method *belief-averaged relative utilitarianism*.

For decision making under risk, that is, when all agents hold the same objective belief, relative utilitarianism is well-known and has been characterized several times (see Karni, 1998; Dhillon, 1998; Dhillon and Mertens, 1999; Segal, 2000; Börgers and Choo, 2017b). For decision making under uncertainty, Gilboa et al. (2004) introduced a restricted Pareto condition that is necessary

and sufficient for the collective belief and utility function to be linear combinations of the agents’ beliefs and utility functions, but allows the weights to vary across profiles. We combine both themes, which leads to a characterization of belief-averaged relative utilitarianism. Our axioms are inspired by those of Dhillon and Mertens and Gilboa et al. and discussed in detail below.

It is instructive to contrast our *ex post* relative utilitarianism with Sprumont’s (2019) *ex ante* relative utilitarianism: the former derives a collective belief and compares acts based on the sum of the agents’ expected utilities under the *collective* belief; the latter first calculates the expected utility of an act for each agent based on their *own* belief and then compares acts by the sum of expected utilities. The upshot is that *ex post* aggregation gives rise to a collective belief and utility function (and thus Savage-type collective preferences), whereas *ex ante* aggregation meets the Pareto principle. Only dictatorships can achieve both as was shown by Mongin (1995). In subsequent work, Mongin argues that preferences based on different beliefs can lead to “spurious unanimities” to which the Pareto condition need not apply. This motivates the restricted Pareto axiom introduced by Gilboa et al. (2004), which *ex post* relative utilitarianism does satisfy and is the most we can hope for given our assumptions about collective preferences. We discuss related literature more extensively in Section 5.

Our two distinguishing axioms are *restricted monotonicity* (or monotonicity for short) and *independence of redundant acts*.

To understand monotonicity, consider the following situation: a group has determined a collective preference (represented by a belief and a utility function). Say they are indifferent between two acts f and g . Now an additional agent who prefers f to g joins the group. Moreover, f induces the same probability distribution over consequences for her belief and the group belief. That is, for every consequence, she assigns the same probability to f resulting in this consequence as does the group. Likewise, g induces the same distribution for both beliefs. Monotonicity demands that the augmented group with the additional agent prefers f to g . In other words, agents get to break ties when joining a group if their belief does not differ from the group’s belief in a way that is relevant for the acts in question. To see why the restriction to conforming beliefs is reasonable, perhaps even desirable, consider two examples:

Example 1. Alice and Bob are planning a weekend activity. They are considering hiking and playing board games at home. There is uncertainty whether it will rain or not, which creates two different states of the world: one in which it will rain and one in which it will not rain. After discussing their beliefs about the weather and their tastes for both activities under different conditions, they decide that they, as a group, are indifferent. Later they receive a call from their friend Charlie, who will join them for the activity. Charlie prefers hiking to playing board games, and so they decide to go hiking.

Since the uncertainty is about the weather and every one of them is unlikely to have access to more relevant information than the others, it is safe to assume that Charlie shares Alice and Bob’s belief. (Indeed, it may well be that all three agents have the same belief.) Hence, Charlie should get to break the tie, so that the group of all three agents prefers hiking to playing board games.

		rain/win	rain/lose	no rain/win	no rain/lose
{Alice, Bob}	belief	10%	40%	10%	40%
	utility from f	0	-1	4	0
Charlie	belief	40%	10%	40%	10%
	utility from f	0	-5	1	0
{Alice, Bob, Charlie}	belief	30%	20%	30%	20%
	utility from f	0	-6	5	0

Table 1: Exemplary numerical values for Example 2. The columns correspond to the four states of the world, resulting from the weather (rain or no rain) and the outcome of the match (win or lose). Each row specifies the belief about states and the utility derived from going to the football match (f) for the corresponding group of agents in each state. We assume playing board games (g) yields utility 0 for all groups in all states. Then Alice and Bob have expected utility 0 for both f and g and are thus indifferent; Charlie’s expected utility for f is -0.1 , so that he prefers g to f . However, the group of all three agents prefers f to g , since they assign utility 0.3 to f .

Example 2. Later Alice sprains her foot, which makes hiking impossible. Alice and Bob bring up going to a football match of their favorite team (f) as an alternative to playing board games (g). Now there is not only uncertainty about the weather but also about whether their team will win or lose the match. As before, they are indifferent between both options. They call Charlie, who prefers playing board games over the match. Nevertheless, the three of them decide to go to the football match.

Now there are four states of the world resulting from two possibilities for the weather and two for the outcome of the match. We can view playing board games as a constant act, one which yields the same consequence in all states since it depends neither on the weather nor on the outcome of the match; but going to the match depends on both factors. In contrast to the belief about the weather, Charlie may well disagree with Alice and Bob’s belief about the game, and so monotonicity does not apply. Say Alice and Bob assign a probability of 50% to rain and of 20% to a win (and both events are independent); Charlie agrees with the probability of rain but assigns 80% to a win. If Charlie is a football expert, the group may assign a large weight to his belief and rate a win at 60% (with again 50% for rain). On the other hand, they assign the same weight to everyone’s utilities. For suitable numerical values (see Table 1), the group decision in Example 2 can occur.

Our definition of monotonicity requires that the set of agents can vary. We will thus consider aggregation functions that take as input the preferences of an arbitrary finite set of agents that may vary in size. When requiring that the preferences of every single-agent group are those of its sole member (which we call faithfulness), restricted monotonicity implies the restricted Pareto condition of Gilboa et al. (2004). It prescribes that whenever all agents are indifferent

between two acts f and g and both induce the same distribution over consequences for every agent's belief, then the group is indifferent between f and g . It is necessary and sufficient for the collective belief and utility function to be linear combinations of the agents' beliefs and utility functions. The weights in both these linear combinations can, however, vary arbitrarily across profiles. Monotonicity implies that the weight of an agent in either linear combination can only depend on her own preferences as we will see. To also eliminate this dependency, we need an axiom that connects profiles with different preferences for the same agent.

Typical candidates are independence axioms, which state that the collective preferences over a set of acts can only depend on the individual preferences over those acts. The most well-known candidate from this family, Arrow's independence of irrelevant alternatives (or acts in our case), demands the above assertion for arbitrary sets of acts. It will lead to an impossibility result even with only mild additional axioms (see, for example, Kalai and Schmeidler (1977) for decision making under risk). Dhillon and Mertens (1999) proposed a weaker version called independence of *redundant* alternatives. It is the blueprint for our *independence of redundant acts*, which requires the above independence for sufficiently large sets of acts. More precisely, say we have two preference profiles (on the same set of agents) and a set of acts \mathcal{G} so that every agent has the same preferences over acts in \mathcal{G} in both profiles and every act is unanimously indifferent to some act in \mathcal{G} in both profiles. Then the collective preferences over acts in \mathcal{G} should be the same in both profiles. The second assumption on \mathcal{G} is equivalent to saying that for both profiles, \mathcal{G} has the same image in utility space as the set of all acts. That is, the vector of expected utilities of every act is equal to the expected utilities of an act in \mathcal{G} .

On top of restricted monotonicity and independence of redundant acts, we make four additional assumptions about the aggregation function: the preferences of any single-agent group are those of its sole member (*faithfulness*), no agent can impose her belief on a group (*no belief imposition*), the preferences of any group depend continuously on the preferences of its members (*continuity*), and relabeling agents does not change the collective preferences (*anonymity*). We show that these six conditions characterize belief-averaged relative utilitarianism.

The proof is modular and yields two intermediary results that are interesting in their own right. First, dropping anonymity, we characterize the class of aggregation functions that assign two positive (and possibly different) weights to every agent, one for her belief and one for her utility function, and then derive the collective preferences of any group from the weighted linear combinations of the beliefs and utility functions of its members. Second, additionally dropping independence of redundant acts allows the weights of an agent to depend on her own preferences. More precisely, the weights of every agent can now be arbitrary continuous and positive functions of her preferences. However, the weights of an agent cannot depend on the other agents' preferences. We give details about the necessity of the axioms for all three results in Section 4.

1. Preferences and Aggregation Functions

Let Ω be a set of states of the world and Σ be a sigma-algebra over Ω . We refer to elements of Σ as events. A probability measure π on (Ω, Σ) is non-atomic if for every $A \in \Sigma$ with $\pi(A) > 0$, there is $B \subset A$ with $0 < \pi(B) < \pi(A)$. We denote by Π the set of all non-atomic and countably additive probability measures. Let X be a set of consequences endowed with a sigma-algebra. An act is a measurable function $f: \Omega \rightarrow X$ that maps states to consequences.

We assume that preferences over acts are the result of maximizing expected utility according to a belief $\pi \in \Pi$ and a utility function $u: X \rightarrow \mathbb{R}$ that is measurable and bounded. We say that π and u represent the preference relation $\succsim \subset \mathcal{F} \times \mathcal{F}$ and conversely that π and u are the belief and the utility function associated with \succsim if

$$f \succsim g \text{ if and only if } \int_{\Omega} (u \circ f) d\pi \geq \int_{\Omega} (u \circ g) d\pi, \quad (\star)$$

for all acts f, g . The strict and symmetric part of \succsim are \succ and \sim , respectively. We denote by $\bar{\mathcal{R}}$ the set of all preference relations that can be represented by expected utility maximization; \mathcal{R} consists of $\bar{\mathcal{R}}$ minus complete indifference, which corresponds to a constant utility function and an arbitrary belief.

All preference relations in \mathcal{R} give rise to a unique belief. Utility functions are only unique up to positive affine transformations. To establish a one-to-one correspondence between preference relations and utility functions, let \mathcal{U} be the set of all non-constant utility functions normalized to the unit interval, that is, $\inf\{u(x): x \in X\} = 0$ and $\sup\{u(x): x \in X\} = 1$; $\bar{\mathcal{U}}$ consists of \mathcal{U} plus the utility function that is constant at 0. When $\pi \in \Pi$ and $u \in \bar{\mathcal{U}}$ represent \succsim , we write $\mathbb{E}_{\succsim}(f) = \int_{\Omega} (u \circ f) d\pi$ for the expected utility of f under π and u .

We postulate an infinite set of potential agents \mathbb{N} . A group I consists of a non-empty and finite subset of agents; the collection of all groups is \mathcal{I} . Symbols in bold face refer to tuples indexed by a set of agents. Every agent has a preference relation $\succsim_i \in \mathcal{R}$. (Notice that no agent may be completely indifferent.) A preference profile $\succsim \in \mathcal{R}^I$ for agents in I specifies the preferences of each agent in I . For $i \in \mathbb{N} - I$ and $\succsim_i \in \mathcal{R}$, we obtain a preference profile \succsim_{+i} for agents in $I \cup \{i\}$ by adding \succsim_i to \succsim . Similarly, when $|I| > 1$ and $i \in I$, \succsim_{-i} is the profile where \succsim_i is deleted. We seek to aggregate the preferences of all members of a group into a collective preference. To this end, we consider an aggregation function Φ that maps every preference profile for every group to an element of $\bar{\mathcal{R}}$.

2. Conditions for Belief-Averaged Relative Utilitarianism

Our axioms for characterizing belief-averaged relative utilitarianism are in part generalizations of Dhillon and Mertens's axioms for relative utilitarianism in the context of risk. The first two, restricted monotonicity and independence of redundant acts, carry the most power in the sense that they rule out other aggregation functions one might come up with.

The restricted monotonicity axiom applies if a group I is indifferent between two acts f and g and is joined by an agent i so that f induces the same distribution over consequences according

to the group's belief and agent i 's belief and so does g . In that case, the augmented group $I \cup \{i\}$ should prefer f to g if and only if i does. Formally, for all $I \in \mathcal{I}$, $i \in \mathbb{N} - I$, $\succsim \in \mathcal{R}^I$, and $\succsim_i \in \mathcal{R}$ with $\succsim = \Phi(\succsim)$ and $\succsim_{+i} = \Phi(\succsim_{+i})$,

$$f \sim g \text{ and } f \succsim_i g \text{ implies } f \succsim_{+i} g \text{ if } \pi \circ f^{-1} = \pi_i \circ f^{-1} \text{ and } \pi \circ g^{-1} = \pi_i \circ g^{-1} \quad (\text{restricted monotonicity})$$

Moreover, a strict preference between f and g for agent i implies a strict collective preference. In the terminology of Gilboa et al. (2004), f and g are lotteries.

The reasoning for independence of redundant acts is that two acts so that all agents are indifferent between them are perfect substitutes for each other, thus making each redundant in the presence of the other. It then prescribes that for two profiles that agree on a set of acts that makes every other act redundant, the collective preferences over this set should also agree. For all $I \in \mathcal{I}$, $\succsim, \succsim' \in \mathcal{R}^I$, and $\mathcal{G} \subset \mathcal{F}$,

$$\begin{aligned} \succsim|_{\mathcal{G}} = \succsim'|_{\mathcal{G}} \text{ implies } \Phi(\succsim)|_{\mathcal{G}} = \Phi(\succsim')|_{\mathcal{G}} \text{ if for all } f \in \mathcal{F} \text{ there are } g, g' \in \mathcal{G} \text{ such that} \\ f \sim_i g \text{ and } f \sim'_i g' \text{ for all } i \in I \quad (\text{independence of redundant acts}) \end{aligned}$$

By the conditions on \mathcal{G} , its image in utility space is the same as that of \mathcal{F} for both profiles \succsim and \succsim' . That is, $\{(\mathbb{E}_{\succsim_i}(g))_{i \in I} : g \in \mathcal{G}\} = \{(\mathbb{E}_{\succsim_i}(f))_{i \in I} : f \in \mathcal{F}\}$ and similarly for \succsim' .

The remaining four axioms are mostly standard. Monotonicity and independence of redundant acts relate different profiles to each other, but do not anchor the aggregation function. For example, there could be a phantom agent with fixed preferences and the collective preferences of every group are derived from belief-averaged relative utilitarianism of the preferences of the group's members and the phantom agent. We can rule out phantom agents by requiring the preferences of a single-agent group to be those of its sole member. For all $i \in \mathbb{N}$ and $\succsim_i \in \mathcal{R}$,

$$\Phi(\succsim_i) = \succsim_i \quad (\text{faithfulness})$$

Part of our definition of monotonicity is that an additional agent can break ties between certain acts in her favor if she has a strict preference. It thus rules out that Φ ignores the utility function of some agent altogether. But so far nothing prevents us from ignoring the beliefs of an agent. To counter this, it suffices that no agent can impose her belief on a group. That is, the belief of a group is not identical to that of one of its members unless the rest of the group would arrive at that belief anyway. Formally, for all $I \in \mathcal{I}$ with $|I| > 1$, $i \in I$, and preference profiles $\succsim \in \mathcal{R}^I$ where $\Phi(\succsim)$ is not complete indifference and π and π' are the beliefs associated with $\Phi(\succsim)$ and $\Phi(\succsim_{-i})$,

$$\pi' \neq \pi_i \text{ implies } \pi \neq \pi_i \quad (\text{no belief imposition})$$

Continuity requires that a small changes in the agents' preferences can only leads to small changes in the collective preferences. To make this precise, we need to equip \mathcal{R} and $\bar{\mathcal{R}}$ with topologies. The uniform metric $\sup\{|\mathbb{E}_{\succsim}(f) - \mathbb{E}_{\succsim'}(f)| : f \in \mathcal{F}\}$ induces a topology on \mathcal{R} . The set of profiles \mathcal{R}^I gets the product topology of \mathcal{R} . The topology on $\bar{\mathcal{R}}$ is that of \mathcal{R} plus the entire

set $\bar{\mathcal{R}}$ (which is thus the only neighborhood of the relation expressing complete indifference). Thus, $\bar{\mathcal{R}}$ is the closure of \mathcal{R} in $\bar{\mathcal{R}}$.

Φ is continuous (continuity)

Lastly, anonymity prescribes that relabeling the agents within a group does not change the collective preferences. For all $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$,

$\Phi(\succsim) = \Phi(\succsim \circ \eta)$ for all permutations η on I (anonymity)

Notice that anonymity as defined here is in general weaker than allowing η to be a bijection between two groups J and I of the same size.

Our results remain true if we require all axioms except restricted monotonicity and faithfulness to hold only for groups of size 2. Independence of redundant acts, continuity, and anonymity are used only for profiles of two agents. The assumption that Φ does not allow belief imposition is used for larger profiles, but it is not hard to check that this could be avoided.

3. Three Characterizations of Aggregation Functions

Conceptually, our most interesting result is a characterization of belief-averaged relative utilitarianism. We will obtain it as a corollary of Theorem 1, which uses the first five axioms (thus excluding anonymity) to characterize affine aggregation functions. These functions assign two positive weights to every agent, one for their belief and one for their utility function, and determine the preferences of every group by adding up the weighted beliefs and utility functions of its members. Importantly, the weights are constant across all profiles, that is, they cannot depend on an agent's own preferences, the preferences of any other agent, or who is a member of the group.

Theorem 1. *Let $|X| \geq 4$ and Φ be an aggregation function. Then the following are equivalent.*

(i) *Φ satisfies restricted monotonicity, independence of redundant acts, faithfulness, no belief imposition, and continuity*

(ii) *There are $\lambda, \mu \in \mathbb{R}_{++}^N$ such that for all $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$, $\Phi(\succsim)$ is represented by $\frac{1}{\sum_{i \in I} \lambda_i} \sum_{i \in I} \lambda_i \pi_i$ and $\sum_{i \in I} \mu_i u_i$*

When additionally requiring Φ to be anonymous, it follows at once that the weights of all agents have to be equal. To see this, consider the two-agent group $I = \{i, j\}$ and any profile $\succsim \in \mathcal{R}^I$ so that the beliefs π_i and π_j and the utility functions u_i and u_j are distinct. Then $\lambda_i \pi_i + \lambda_j \pi_j \neq \lambda_i \pi_j + \lambda_j \pi_i$ whenever $\lambda_i \neq \lambda_j$. Likewise, $\mu_i u_i + \mu_j u_j \neq \mu_i u_j + \mu_j u_i$ if $\mu_i \neq \mu_j$. Thus, anonymity can only hold if $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$. Since multiplication of all weights by the same positive constant does not change the collective preferences, we may assume that all weights are equal to 1. This gives a characterization of belief-averaged relative utilitarianism as the only aggregation function that satisfies all our axioms. It derives the collective preferences by averaging the beliefs and adding up the normalized utility functions of all agents.

Corollary 1. *Let $|X| \geq 4$ and Φ be an aggregation function. Then the following are equivalent.*

(i) *Φ satisfies restricted monotonicity, independence of redundant acts, faithfulness, no belief imposition, continuity, and anonymity*

(ii) *For all $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$, $\Phi(\succsim)$ is represented by $\frac{1}{|I|} \sum_{i \in I} \pi_i$ and $\sum_{i \in I} u_i$*

The proof of Theorem 1 proceeds as follows. First, we only consider the implications of restricted monotonicity, faithfulness, and no belief imposition. The former two imply the restricted Pareto condition and thus that beliefs and utility functions are aggregated linearly (with positive weights for utility functions by the strict part of monotonicity). The additional strength of monotonicity lies in the fact that if an agent joins a group, the belief of the augmented group is an affine combination of the belief of the original group and that of the agent. Assuming all beliefs are affinely independent, it follows that the relative weights of the agents in the original group cannot change. The analogous statement holds for utility functions.

Now, given any profile, we can apply this conclusion to every agent and the subprofile excluding this agent. It follows that the magnitude of the weight of the belief and the utility function of an agent can only depend on her own preferences. The signs may however depend on the preferences of the other agents. Since weights for utility functions have to be positive, any dependence on other agents' preferences vanishes for the weights of utility functions. Surprisingly, continuity allows us to get the same conclusion for beliefs. If the weight for an agent's belief ever were to change sign, by continuity, her weight would have to be zero in some (two-agent) profile. But then the second agent would get to impose her belief, which is ruled out. We conclude that the weight for an agent's belief cannot change sign. If it were to be negative regardless of her preferences, we could find a profile where the collective belief assigns a negative probability to some event, which gives a contradiction.

In summary, we derive the following intermediate result, which is interesting in its own right.

Proposition 1. *Let $|X| \geq 4$ and Φ be an aggregation function. Then the following are equivalent.*

(i) *Φ satisfies restricted monotonicity, faithfulness, no belief imposition, and continuity*

(ii) *There are continuous functions $\lambda, \mu: \mathcal{R} \rightarrow \mathbb{R}_{++}^{\mathbb{N}}$ such that for all $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$, $\Phi(\succsim)$ is represented by $\frac{1}{\sum_{i \in I} \lambda_i(\succsim_i)} \sum_{i \in I} \lambda_i(\succsim_i) \pi_i$ and $\sum_{i \in I} \mu_i(\succsim_i) u_i$*

The last step is to show that additionally presuming independence of redundant acts gives that the weights of an agent have to be constant, that is, that the functions λ_i and μ_i in Proposition 1 are constant. By applying independence of redundant acts to a suitable two-agent profile, it follows relatively quickly that the weights of an agent cannot depend on her belief. To conclude they are independent of her utility function as well, we consider a two-agent profile in which every act is unanimously indifferent to an act with a range of only three consequences $\{x_0, x_1, x^*\}$. If the focal agent changes her utility for other consequences in a way that does not change the image of the profile in utility space, we can apply independence of redundant acts and conclude that the collective preferences over acts with range $\{x_0, x_1, x^*\}$

do not change. This can only be if the weights of that agent remain the same. By repeatedly applying this assertion, we can construct a path between any pair of utility functions so that neighboring utility functions result in the same weights.

4. Necessity of the Axioms

We discuss the necessity of the axioms for Corollary 1 first since the examples we give here will work for Theorem 1 and Proposition 1 as well.

Not much can be said about the aggregation function if it does not satisfy restricted monotonicity. For example, every faithful function that is constant on profiles with two or more agents satisfies all axioms except monotonicity. Here is an example, adapted from Dhillon and Mertens (1999), that violates monotonicity but satisfies the restricted Pareto condition. The collective belief is the average of the agents' beliefs. Consider the closure of the profile in utility space and let $(u^i)_{i \in I}$ be the unique point that maximizes the product of utilities. Let the collective utility function be the linear combination of the agents' utility functions where the weight of agent i is $\prod_{j \in I \setminus \{i\}} u^j$. Since the weights for an agent are the same in any two profiles with the same image in utility space, this function satisfies independence of redundant acts.

The class of functions which satisfy all axioms but independence of redundant acts and anonymity is characterized by Proposition 1. Anonymity holds if and only if the weight functions λ_i and μ_i are the same across agents.

Without faithfulness, we could have a “phantom agent” whose belief and utility function are always added on top of the beliefs and utility functions of real agents with a constant weight.

If Φ is not continuous, agents with the same preferences could be handled specially. For example, let (α_n) be a strictly increasing positive sequence. Then if $\Phi(\succsim)$ is represented by $\frac{1}{\sum_{\succsim \in \mathcal{R}} \alpha_{n(\succsim)}} \sum_{\succsim \in \mathcal{R}} \alpha_{n(\succsim)} \pi_i$ and $\sum_{\succsim \in \mathcal{R}} \alpha_{n(\succsim)} u_i$, where $n(\succsim)$ is the number of agents in the profile \succsim with preferences \succsim , then Φ satisfies all axioms but continuity.

It is open whether no belief imposition is necessary for the conclusion of Corollary 1. In the absence of anonymity, that is, for Theorem 1, it is necessary, however. In that case, we lose the decomposable form for the group belief if Φ allows belief imposition. For example, we could have that the group belief is π_1 whenever agent 1 is part of the group and $\frac{1}{|I|} \sum_{i \in I} \pi_i$ otherwise. Thus, whether the belief of an agent gets non-zero weight could depend on whether some particular other agent is present.

For Proposition 1, continuity is even more vital. Without it, the weight of an agent's belief can even be negative. Consider $\Omega = [0, 1]$ equipped with the Borel sigma-algebra. Let $\tilde{\pi}$ be the uniform distribution on Ω and, for a non-atomic measure π on Ω , let $\rho(\pi) = \sup\{\frac{\pi(E)}{\tilde{\pi}(E)} : E \in \Sigma\}$. (Since non-atomic measures have continuous density functions, $\rho(\pi)$ is finite.) For $i \geq 2$, let $\lambda_i(\succsim_i) = \frac{1}{3^i \rho(\pi_i)}$, and λ_1 as well as all μ_i be constant at 1. If $\Phi(\succsim)$ is represented as in Proposition 1 except that the collective belief is $\pi_1 - \sum_{i \in I - \{1\}} \lambda_i(\succsim_i) \pi_i$ whenever $1 \in I$ and $\pi_1 = \tilde{\pi}$, it satisfies all axioms but continuity.

Our proof requires $|X| \geq 4$ since it relies on profiles with three linearly independent utility functions. We do not know if our results hold when $|X| = 3$.

5. Relationship to the Literature on Preference Aggregation Under Uncertainty and Risk

Much of the early literature on group decision making under uncertainty has focused on the Pareto condition, which requires that a unanimous preference among agents prevails in the collective preferences. For decision making under risk, that is, when all agents have the same belief and acts become lotteries, Harsanyi’s (1955) well-known theorem shows that under the Pareto condition, the utility function of a group has to be a linear combination of the individual utility functions. Hylland and Zeckhauser (1979) consider this condition in a multi-profile framework under uncertainty and show that it is incompatible with non-dictatorship and separate aggregation of beliefs and utility functions. Mongin (1995) examines the implications of different degrees of the Pareto condition for single profiles when individual and group preferences are of Savage-type as in the present paper. Roughly, he finds that it implies the existence of a dictator for all but degenerate profiles. Mongin (1998) shows that this result persists in Anscombe and Aumann’s (1963) model of subjective expected utility. It disappears if one allows utility functions to be state-dependent, but reappears for intermediary preference models, which allow identification of beliefs.

In contrast to these negative findings, Gilboa et al. (2004) show that the previously mentioned restricted Pareto condition is equivalent to linear aggregation of beliefs and utility functions. In particular, the restricted Pareto condition does not apply to what Mongin (2016) calls “spurious unanimities”. Gilboa et al. (2014) consider an intermediate between the full and the restricted Pareto condition, which they call no-betting-Pareto dominance. It states that one act dominates another if it Pareto dominates it in the usual sense and there is a belief such that if all agents held it, they would also unanimously prefer the first act to the second. They argue that no-betting-Pareto dominance characterizes situations in which agents can benefit from trade, but do not seek to determine how it restricts preference aggregation.

Among the few multi-profile approaches is that of Dietrich (2019), which adds consistency with Bayesian updating and continuity of the aggregation function to the restricted Pareto condition. These conditions imply that utilities are aggregated linearly and beliefs are aggregated *geometrically*, that is, the group belief is a geometric mean of the individual beliefs. Moreover, the weight of an agent in either of these combinations can depend on the profile of utility functions, but not on beliefs.

Several authors dispense with the assumption that collective preferences are based on subjective expected utility maximization. Most relevantly, Sprumont (2019) characterizes a different form of relative utilitarianism where acts are compared by their cumulative expected utility $\sum_{i \in I} \mathbb{E}_{\pi_i}(f) = \sum_{i \in I} \int_{\Omega} (u_i \circ f) d\pi_i$. By contrast, belief-averaged relative utilitarianism yields the order induced by $\int_{\Omega} (\sum_{i \in I} u_i \circ f) d\pi$, where $\pi = \frac{1}{|I|} \sum_{i \in I} \pi_i$. Thus, the difference is whether the expectation is taken before or after summing over agents. For his characterization, Sprumont assumes the full Pareto axiom, independence of inessential expansions (a strengthening of independence of redundant acts),¹ belief irrelevance (the ranking of constant acts is indepen-

¹More precisely, independence of inessential expansions requires that if two profiles agree on a set of acts \mathcal{G} so

dent of beliefs), and that collective preferences are continuous and satisfy Savage’s sure-thing principle. As a high-level summary, one could say that the assumptions about the aggregation function are stronger whereas those about collective preferences are weaker.

Alon and Gayer (2016) assume that groups have Gilboa and Schmeidler’s (1989) max-min expected utility preferences, where acts are compared based on their minimal expected utility within a set of beliefs. In addition to the restricted Pareto condition, they assume that if all agents believe that one event is more likely than another, then so does the group. These two axioms imply that utility functions are aggregated linearly and the set of group beliefs consists of convex combinations of individual beliefs.

Nascimento (2012) studies the aggregation of the preferences of experts that agree on the ranking of risky prospects, but are otherwise very general; in particular, they need not result from subjective expected utility maximization. He gives a set of assumptions about the experts’ and the aggregated preferences under which the latter are the result of a compromise between utilitarian aggregation and the Rawlsian criterion.

Some results from the literature on preference aggregation under risk are particularly relevant. First, Dhillon and Mertens’s (1999) characterization of relative utilitarianism, which derives the collective preference by adding up the normalized utility functions of the agents. As Harsanyi, they assume that individual and collective preferences are given by linear utility functions on lotteries. They require the aggregation function to satisfy continuity, anonymity, a monotonicity condition, and independence of redundant alternatives.² Monotonicity and independence of redundant alternatives are analogs of restricted monotonicity and independence of redundant acts. In the same vein, Dhillon (1998) characterizes relative utilitarianism with a Pareto condition for groups of agents instead of monotonicity. It requires that if two disjoint groups agree on the preference between two lotteries, then the union of both groups entertains the same preference. Like our monotonicity condition, it acts on variable sets of agents. Börgers and Choo (2017a) discovered a flaw in Dhillon’s proof, which presumably stems from the fact that she allows agents to have equal or opposite utility functions without assuming continuity. Börgers and Choo (2017b) provide a more accessible characterization of relative utilitarianism along the same lines as Dhillon and Mertens.

6. Conclusions and Open Problems

We have shown that restricted monotonicity and independence of redundant acts in conjunction with three axioms which require the aggregation function to be well-behaved, that is, faithful, continuous, and free from belief imposition, necessitate affine aggregation of beliefs and utility functions with constant weights. If anonymity holds in addition, all weights have to be equal and belief-averaged relative utilitarianism remains as the only possibility. These results are based on a characterization of aggregation functions that satisfy monotonicity but not necessarily

that \mathcal{G} contains a most-preferred and a least-preferred act for every agent, then the collective preferences over acts in \mathcal{G} are the same for both profiles.

²Dhillon and Mertens (1999) use a weaker monotonicity condition, but one which also allows them to prove the equivalent of our Lemma 5.

independence of redundant acts. We have thus drawn a clear picture of monotonic aggregation functions.

We leave open if there is a succinct representation of the class of functions that satisfy independence of redundant acts (plus perhaps some standard axioms). In particular, it is unclear how much independence of redundant acts restricts the aggregation of beliefs. Finally, it would be desirable to determine whether ruling out belief imposition is necessary for the conclusion of Corollary 1.

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APPENDIX: Proofs

An aggregation function Φ gives rise to two functions ϕ and ψ , which take a profile $\succsim \in \mathcal{R}^I$ for an arbitrary group I to a collective belief $\phi(\succsim) \in \Pi$ and a collective utility function $\psi(\succsim) \in \bar{\mathcal{U}}$. Whenever the collective utility function is trivial, that is, $\psi(\succsim) = 0$, $\phi(\succsim)$ is not uniquely determined. In fact, it can be arbitrary.

Preferences are invariant under multiplication of the belief by a positive constant and positive affine transformations of utility functions. For measures π, π' on Ω and utility functions u, u' on X , we write $\pi \equiv \pi'$ if $\pi = \alpha\pi'$ for some $\alpha > 0$ and $u \equiv u'$ if $u = \alpha u' + \beta$ for $\alpha > 0$ and $\beta \in \mathbb{R}$.

Theorem 1 requires that we find $\lambda, \mu \in R_{++}^{\mathbb{N}}$ such that $\phi(\succsim) \equiv \sum_{i \in I} \lambda_i \pi_i$ and $\psi(\succsim) \equiv \sum_{i \in I} \mu_i u_i$ for all $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$. The proof proceeds in three main steps. First, we examine the implications of restricted monotonicity in conjunction with faithfulness and no belief imposition. These axioms imply that in almost all profiles, the weights assigned to an agent’s belief and utility function can only depend on her own preferences. The weights for beliefs may be negative however. Second, we add continuity, which allows us to rule out

negative weights and to extend the obtained representation to all profiles (Proposition 1). Lastly, independence of redundant acts implies that the weights of an agent cannot depend on her own preferences either and thus have to be constant across all profiles.

A. Implications of Restricted Monotonicity

The proofs that the weight of the belief and the weight of the utility function of an agent can only depend on her own preferences in Sections A.1 and A.2 proceed along the same lines. For the most part, the proof for beliefs requires more work, since we cannot rule out negative weights. Thus, we advise readers interested in the proofs to take a look at Appendix A.2 first.

A.1. Aggregation of Beliefs

The first lemma is the basis from which we will derive all further conclusions about belief aggregation. It states that if an agent joins a group, the new group belief is an affine combination of the previous group belief and the belief of the agent. The restricted Pareto condition of Gilboa et al. (2004) already implies that the collective belief is an affine combination of its members' beliefs. The additional strength of this conclusion lies in the fact that no matter which belief the new agent holds, it is always combined with the same belief of the original group. If we assume that an agent cannot impose her belief on the group, her weight in the affine combination cannot be 1.

Lemma 1. *Assume that Φ satisfies restricted monotonicity and rules out belief imposition. Let $I \in \mathcal{I}$, $i \in I$, and $\succsim \in \mathcal{R}^I$ with $\psi(\succsim) \neq 0$. Then, $\phi(\succsim) = (1 - \alpha)\phi(\succsim_{-i}) + \alpha\pi_i$ for some $\alpha \in \mathbb{R} - \{1\}$.*

Proof. Let $\succsim = \Phi(\succsim)$ and $\succsim_{-i} = \Phi(\succsim_{-i})$. Monotonicity implies that $f \sim g$ whenever $f \sim_{-i} g$, $f \sim_i g$ and $\phi(\succsim_{-i}) \circ f^{-1} = \pi_i \circ f^{-1}$ and $\phi(\succsim_{-i}) \circ g^{-1} = \pi_i \circ g^{-1}$. Thus, (the two-agent case of) Theorem 1 of Gilboa et al. (2004) implies that $\phi(\succsim) = (1 - \alpha)\phi(\succsim_{-i}) + \alpha\pi_i$ for some $\alpha \in \mathbb{R}$. If $\pi_i = \phi(\succsim_{-i})$, we can choose α arbitrarily. Otherwise, $\pi_i \neq \phi(\succsim)$, since Φ rules out belief imposition, and so $\alpha \neq 1$. \square

Lemma 2. *Assume that Φ satisfies restricted monotonicity and faithfulness and rules out belief imposition. Let $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$ with $\psi(\succsim) \neq 0$. Then $\phi(\succsim) = \sum_{i \in I} \lambda_i \pi_i$ for some $\lambda \in \mathbb{R}^I$ with $\sum_{i \in I} \lambda_i = 1$. Moreover, if $(\pi_i)_{i \in I}$ are affinely independent, then $\lambda \in (\mathbb{R} - \{0\})^I$ and λ is unique.*

Proof. Since Φ is faithful, we have that $\phi(\succsim_1) = \pi_1$. Now let one agent after another join. We apply Lemma 1 at each step and get $\phi(\succsim) = \sum_{i \in I} \lambda_i \pi_i$ for some $\lambda \in \mathbb{R}^I$ with $\sum_{i \in I} \lambda_i = 1$.

If $(\pi_i)_{i \in I}$ are affinely independent, λ is unique. We prove by induction over $|I|$ that $\lambda_i \neq 0$ for all i . If $|I| = 1$, then $\lambda = 1$ is forced. Now suppose that $|I| > 1$ and let $i, j \in I$. By the induction hypothesis, we have $\phi(\succsim_{-i}) = \sum_{k \in I - \{i\}} \lambda'_k \pi_k$ and $\phi(\succsim_{-j}) = \sum_{k \in I - \{j\}} \lambda''_k \pi_k$ for some $\lambda' \in (\mathbb{R} - \{0\})^{I - \{i\}}$ and $\lambda'' \in (\mathbb{R} - \{0\})^{I - \{j\}}$. Lemma 1 implies that

$$\phi(\succsim) = (1 - \alpha)\phi(\succsim_{-i}) + \alpha\pi_i = (1 - \beta)\phi(\succsim_{-j}) + \beta\pi_j$$

for some $\alpha, \beta \in \mathbb{R} - \{1\}$. We set $\boldsymbol{\lambda} = ((1 - \alpha)\boldsymbol{\lambda}', \alpha) = ((1 - \beta)\boldsymbol{\lambda}'', \beta)$, where α and β appear in position i and j respectively. Since $\alpha \neq 1$ and $\lambda'_k \neq 0$, it follows that $\lambda_k \neq 0$ for all $k \in I - \{i\}$. Similarly, $\beta \neq 1$ implies that $\lambda_k \neq 0$ for all $k \in I - \{j\}$. \square

We define the dimension of a vector of beliefs $\boldsymbol{\pi} \in \Pi^I$ as the maximal number of affinely independent probability distributions in $\{\pi_i : i \in I\}$. Equivalently, the dimension of $\boldsymbol{\pi}$ is the dimension of the subset $\{(\pi_i(E))_{i \in I} : E \in \Sigma\}$ of \mathbb{R}^I . For later use, we prove a fact for $\boldsymbol{\pi}$ with dimension at least 3.

Lemma 3. *Assume that Φ satisfies restricted monotonicity and rules out belief imposition. Let $I \in \mathcal{I}$ and $\succ \in \mathcal{R}^I$ with $\psi(\succ) \neq 0$. If $\boldsymbol{\pi}$ has dimension at least 3, there are distinct $i, j \in I$ such that $\phi(\succ_{-i,j})$, π_i , and π_j are affinely independent.*

Proof. Suppose $\{1, 2, 3\} \subset I$. Since $\boldsymbol{\pi}$ has dimension at least 3, we may assume that π_1, π_2 , and π_3 are affinely independent. So $\phi(\succ)$ cannot be in the affine hull of all three pairs from $\{\pi_1, \pi_2, \pi_3\}$, for if say $\phi(\succ)$ is in the affine hull of $\{\pi_1, \pi_2\}$ and $\{\pi_1, \pi_3\}$, then $\phi(\succ) = \pi_1$ and so is not in the affine hull of $\{\pi_2, \pi_3\}$. Assume that $\phi(\succ)$ is not in the affine hull of $\{\pi_1, \pi_2\}$. Then Lemma 1 implies that $\phi(\succ_{-1,2})$ is not in the affine hull of $\{\pi_1, \pi_2\}$. Since $\pi_1 \neq \pi_2$, $\phi(\succ_{-1,2})$, π_1 , and π_2 are affinely independent. \square

Lemma 2 ensures that the group belief is always an affine combination of the agents' beliefs. To show that ϕ has the form claimed in Theorem 1, we have to prove that the relative weight of an agent in this affine combination depends only on her own belief and utility function. For now, we have to be content with a weaker conclusion, which allows negative weights. For the rest of this section, we will assume that *beliefs and utility functions are pairwise distinct in all profiles*.

Lemma 4. *Assume that Φ satisfies restricted monotonicity and faithfulness and rules out belief imposition. Then there are $\boldsymbol{\lambda} : \mathcal{R} \rightarrow (\mathbb{R} - \{0\})^{\mathbb{N}}$ and for all $I \in \mathcal{I}$, $\boldsymbol{\sigma}^I : \mathcal{R}^I \rightarrow \{-1, 1\}^I$ such that for all $I \in \mathcal{I}$ and $\succ \in \mathcal{R}^I$, $\phi(\succ) \equiv \sum_{i \in I} \sigma_i^I(\succ) \lambda_i(\succ_i) \pi_i$. Moreover, $\frac{\sigma_i^I(\succ)}{\sigma_j^I(\succ)} = \frac{\sigma_i^{\{i,j\}}(\succ_i, \succ_j)}{\sigma_j^{\{i,j\}}(\succ_i, \succ_j)}$ for all I , $i, j \in I$, and $\succ \in \mathcal{R}^I$.*

Proof. For $l \in \mathbb{R} - \{1, 2\}$, let $I_l = \{1, 2, l\}$ and $\mathcal{R}_l \subset \mathcal{R}^{I_l}$ be the set of all profiles for agents I_l such that π_1, π_2 , and π_l are affinely independent. Let $l \in \mathbb{N} - \{1, 2\}$ be arbitrary and fix some $\tilde{\succ} \in \mathcal{R}_l$. By Lemma 2, there is a unique function $\boldsymbol{\kappa} : \mathcal{R}_l \rightarrow (\mathbb{R} - \{0\})^{I_l}$ such that $\phi(\succ) = \sum_{i \in I_l} \kappa_i(\succ) \pi_i$ for all $\succ \in \mathcal{R}_l$. For $i, j \in I_l$ and $\succ \in \mathcal{R}_l$, let $\lambda^{i,j}(\succ) = \frac{\kappa_j(\succ_{-j, \tilde{\succ}_j}) \kappa_i(\succ)}{\kappa_j(\succ) \kappa_i(\succ_{-j, \tilde{\succ}_j})}$. The fact that $\boldsymbol{\kappa}$ maps to $(\mathbb{R} - \{0\})^{I_l}$ ensures that $\lambda^{i,j}$ is well-defined. Then, let $\lambda_i(\succ_i) = \frac{|\kappa_i(\tilde{\succ}_i)|}{|\lambda^{i,j}(\tilde{\succ})|}$ and $\sigma_i^{I_l}(\succ) = \text{sign}(\kappa_i(\succ))$.

We show that λ_i is independent of j and \succ_{-i} and thus well-defined. We proceed in three steps. Before, note that the projection of \mathcal{R}_l to \mathcal{R} that returns the preferences of i is onto, and so λ_i is a function on all of \mathcal{R} .

Step 1. Let $k \in I_l - \{i, j\}$. We show that $\frac{\kappa_i(\mathfrak{A})}{\kappa_j(\mathfrak{A})}$ is independent of \mathfrak{A}_k . To this end, let $\mathfrak{A}' \in \mathcal{R}_l$ such that $\mathfrak{A}'_{-k} = \mathfrak{A}_{-k}$. By Lemma 1, we have that

$$\begin{aligned}\phi(\mathfrak{A}) &= (1 - \alpha)\phi(\mathfrak{A}_{-k}) + \alpha\pi_k = \kappa_i(\mathfrak{A})\pi_i + \kappa_j(\mathfrak{A})\pi_j + \kappa_k(\mathfrak{A})\pi_k, \text{ and} \\ \phi(\mathfrak{A}') &= (1 - \beta)\phi(\mathfrak{A}'_{-k}) + \beta\pi'_k = \kappa_i(\mathfrak{A}')\pi_i + \kappa_j(\mathfrak{A}')\pi_j + \kappa_k(\mathfrak{A}')\pi'_k,\end{aligned}$$

for some $\alpha, \beta \in \mathbb{R} - \{1\}$. Affine independence of π_1, π_2, π_l and π_1, π_2, π'_l implies that $\kappa_i(\mathfrak{A})\pi_i + \kappa_j(\mathfrak{A})\pi_j \equiv \phi(\mathfrak{A}_{-k}) = \phi(\mathfrak{A}'_{-k}) \equiv \kappa_i(\mathfrak{A}')\pi_i + \kappa_j(\mathfrak{A}')\pi_j$. In particular, $\frac{\kappa_i(\mathfrak{A})}{\kappa_j(\mathfrak{A})} = \frac{\kappa_i(\mathfrak{A}')}{\kappa_j(\mathfrak{A}')}$. Repeated application yields the desired independence.

Step 2. We show that $\lambda^{i,j}(\mathfrak{A})$ is independent of i and j . This is tedious, but only uses Step 1. Let $k \in I_l - \{i, j\}$. First we show independence of j .

$$\begin{aligned}\lambda^{i,j}(\mathfrak{A}) &= \frac{\kappa_j(\mathfrak{A}_{-j}, \tilde{\mathfrak{A}}_j)\kappa_i(\mathfrak{A})}{\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-j}, \tilde{\mathfrak{A}}_j)} \\ &= \frac{\kappa_j(\mathfrak{A}_{-j,k}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_j)\kappa_i(\mathfrak{A})}{\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-j,k}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_j)} \\ &= \frac{\kappa_j(\mathfrak{A}_{-j,k}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_j)\kappa_i(\mathfrak{A})\kappa_k(\mathfrak{A}_{-j,k}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_j)}{\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-j,k}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_j)\kappa_k(\mathfrak{A}_{-j,k}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_j)} \\ &= \frac{\kappa_j(\tilde{\mathfrak{A}})\kappa_k(\mathfrak{A}_{-k}, \tilde{\mathfrak{A}}_k)\kappa_i(\mathfrak{A})}{\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-k}, \tilde{\mathfrak{A}}_k)\kappa_k(\tilde{\mathfrak{A}})} = \lambda^{i,k}(\mathfrak{A})\end{aligned}$$

Verifying independence of i is very similar.

$$\begin{aligned}\lambda^{i,j}(\mathfrak{A}) &= \frac{\kappa_j(\mathfrak{A}_{-j}, \tilde{\mathfrak{A}}_j)\kappa_i(\mathfrak{A})}{\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-j}, \tilde{\mathfrak{A}}_j)} \\ &= \frac{\kappa_j(\mathfrak{A}_{-k,i}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_i)\kappa_i(\mathfrak{A})}{\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-k,i}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_i)} \\ &= \frac{\kappa_j(\mathfrak{A}_{-k,i}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_i)\kappa_i(\mathfrak{A})\kappa_k(\mathfrak{A}_{-k,i}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_i)}{\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-k,i}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_i)\kappa_k(\mathfrak{A}_{-k,i}, \tilde{\mathfrak{A}}_k, \tilde{\mathfrak{A}}_i)} \\ &= \frac{\kappa_j(\mathfrak{A}_{-j}, \tilde{\mathfrak{A}}_j)\kappa_i(\mathfrak{A})\kappa_k(\mathfrak{A})}{\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A})\kappa_k(\mathfrak{A}_{-j}, \tilde{\mathfrak{A}}_j)} = \lambda^{k,j}(\mathfrak{A})\end{aligned}$$

Step 3. We show that $\lambda_i(\mathfrak{A})$ is independent of \mathfrak{A}_{-i} and j .

$$\lambda_i(\mathfrak{A}) = \frac{|\kappa_i(\mathfrak{A})|}{|\lambda^{i,j}(\mathfrak{A})|} = \frac{|\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-j}, \tilde{\mathfrak{A}}_j)|}{|\kappa_j(\mathfrak{A}_{-j}, \tilde{\mathfrak{A}}_j)|} = \frac{|\kappa_j(\tilde{\mathfrak{A}})\kappa_i(\mathfrak{A}_{-i}, \tilde{\mathfrak{A}}_i)|}{|\kappa_j(\mathfrak{A}_{-i}, \tilde{\mathfrak{A}}_i)|},$$

where we use Step 1 for the last equality. The resulting term is independent of \mathfrak{A}_{-i} and, by Step 2, of j .

Now it is easy to see that

$$\phi(\mathfrak{A}) = \sum_{i \in I_l} \kappa_i(\mathfrak{A})\pi_i \equiv \sum_{i \in I_l} \frac{\kappa_i(\mathfrak{A})}{|\lambda^{i,j_i}(\mathfrak{A})|} \pi_i = \sum_{i \in I_l} \sigma_i^{I_l}(\mathfrak{A}) \lambda_i(\mathfrak{A}) \pi_i,$$

where $j_i \in I_l - \{i\}$ for all i . For the second equality, we used the fact that $\lambda^{i,j}$ is independent of i and j .

Since l was arbitrary, we have now defined λ_i for each $i \in \mathbb{N}$. However, we have defined λ_1 and λ_2 multiple times, once for each $l \in \mathbb{N} - \{1, 2\}$. So we have to check that these definitions are not conflicting. It follows from Lemma 1 that the ratio between λ_1 and λ_2 is the same for each triple $\{1, 2, l\}$. Thus, we can define λ_1 and λ_2 as obtained for, say, $l = 3$ and scale the triples $(\lambda_1, \lambda_2, \lambda_l)$ obtained for the remaining l appropriately.

Step 4. Now we define σ for the remaining profiles. Our strategy will be to first define it for two-agent profiles, then inductively for all profiles such that π has dimension at least 3, and then for the remaining profiles. At each point, we maintain that $\frac{\sigma_i^I(\succ)}{\sigma_j^I(\succ)} = \frac{\sigma_i^{\{i,j\}}(\tilde{\succ}_i, \tilde{\succ}_j)}{\sigma_j^{\{i,j\}}(\tilde{\succ}_i, \tilde{\succ}_j)}$, which we will refer to as the ratio condition on σ . We omit the superscript in expressions like $\sigma_i^I(\succ)$ from now on, since it is clear from the profile.

Let $I \in \mathcal{I}$ and $\succ \in \mathcal{R}^I$. If $|I| = 2$, say $I = \{1, 2\}$, then $\phi(\succ) = \alpha_1\pi_1 + \alpha_2\pi_2$ for a unique $\alpha \in (\mathbb{R} - \{0\})^I$. We define $\sigma_i(\succ) = \text{sign}(\alpha_i)$ for $i \in I$.

Now assume that $|I| \geq 3$ and π has dimension at least 3; assume further that we have defined σ for all profiles of dimension three on fewer than $|I|$ agents such that the ratio condition holds. Let $i \in I$ such that $\pi_i \neq \phi(\succ_{-i})$, which exists by Lemma 3. We show that there is $s \in \{-1, 1\}$ such that $\frac{s}{\sigma_j(\succ_{-i})} = \frac{\sigma_i(\tilde{\succ}_i, \tilde{\succ}_j)}{\sigma_j(\tilde{\succ}_i, \tilde{\succ}_j)}$ for all $j \in I - \{i\}$. If not, there are j, k which require $s = 1$ and $s = -1$ respectively. It is not hard to see that then there must be j, k with this property such that (π_i, π_j, π_k) has dimension 3. Then we have

$$\frac{\sigma_j(\succ_{-i})}{\sigma_k(\succ_{-i})} = \frac{\sigma_j(\tilde{\succ}_j, \tilde{\succ}_k)}{\sigma_k(\tilde{\succ}_j, \tilde{\succ}_k)} \neq \frac{\sigma_j(\tilde{\succ}_i, \tilde{\succ}_j)}{\sigma_i(\tilde{\succ}_i, \tilde{\succ}_j)} \frac{\sigma_i(\tilde{\succ}_i, \tilde{\succ}_k)}{\sigma_k(\tilde{\succ}_i, \tilde{\succ}_k)} = \frac{\sigma_j(\succ_{i,j,k})}{\sigma_i(\succ_{i,j,k})} \frac{\sigma_i(\succ_{i,j,k})}{\sigma_k(\succ_{i,j,k})} = \frac{\sigma_j(\succ_{i,j,k})}{\sigma_k(\succ_{i,j,k})}$$

where we use the fact that $\sigma^{I-\{i\}}$ satisfies the ratio condition for the first equality. This is a contradiction, since $\sigma^{\{i,j,k\}}$ also satisfies the ratio condition. Thus we can find s as required.

Lemma 1 implies that $\phi(\succ) = (1 - \alpha)\phi(\succ_{-i}) + \alpha\pi_i$ for some unique $\alpha \in \mathbb{R} - \{1\}$. If $\alpha < 1$, we set $\sigma^I = (\sigma^{I-\{i\}}, s)$; if $\alpha > 1$, set $\sigma^I = -(\sigma^{I-\{i\}}, s)$.

Lastly, if π has dimension 2, let $i \in \mathbb{N} - I$ and consider a profile \succ' for agents in $I \cup \{i\}$ such that $\succ_{-i}' = \succ$ and π' has dimension 3. By Lemma 2, $\pi'_i \neq \phi(\succ)$ and so $\phi(\succ') = (1 - \alpha)\phi(\succ) + \alpha\pi'_i$ for some $\alpha \neq 1$. If $\alpha < 1$, we set $\sigma_j(\succ) = \sigma_j(\succ')$ for all $j \in I$; if $\alpha > 1$, set $\sigma_j(\succ) = -\sigma_j(\succ')$.

We still have to make sure that these definitions of λ and σ are consistent with ϕ . For $I \in \mathcal{I}$ and $\succ \in \mathcal{R}^I$, let $\bar{\phi}(\succ) \equiv \sum_{i \in I} \sigma_i^I(\succ) \lambda_i(\succ_i) \pi_i$. We show by induction over $|I|$ that ϕ and $\bar{\phi}$ agree on all profiles \succ . First we assume that π has dimension at least 3. Later we will take care of the remaining profiles.

We start with two observations.

Step 5. Let \succ be a profile for agents in I such that π has dimension 3 and π_{-i} has dimension 2. If ϕ and $\bar{\phi}$ agree on \succ , then they also agree on \succ_{-i} . By Lemma 1 and the assumption, we have

$$\phi(\succ) = (1 - \alpha)\phi(\succ_{-i}) + \alpha\pi_i \equiv \sum_{j \in I - \{i\}} \sigma_j(\succ) \lambda_j(\succ_j) \pi_j + \sigma_i(\succ) \lambda_i(\succ_i) \pi_i$$

for some $\alpha \in \mathbb{R} - \{1\}$. Since π_i is not in the affine hull of $(\pi_j)_{j \in I - \{i\}}$, we have to have $(1 - \alpha)\phi(\succ_{-i}) \equiv \sum_{j \in I - \{i\}} \sigma_j(\succ) \lambda_j(\succ_j) \pi_j$. If $\alpha < 1$, then by definition, $\sigma_j(\succ_{-i}) = \sigma_j(\succ)$

for all $j \in I - \{i\}$, and so $\phi(\succ_{-i}) \equiv \sum_{j \in I - \{i\}} \sigma_j(\succ_{-i}) \lambda_j(\succ_j) \pi_j \equiv \bar{\phi}(\succ_{-i})$. If $\alpha > 1$, then $\sigma_j(\succ_{-i}) = -\sigma_j(\succ)$ for all j , and again $\phi(\succ_{-i}) = \bar{\phi}(\succ_{-i})$ follows.

Step 6. Let \succ be a profile for agents in I and $i, j \in I$. If ϕ and $\bar{\phi}$ agree on \succ_{-i} and \succ_{-j} and $\bar{\phi}(\succ_{-i,j}), \pi_i$, and π_j are affinely independent, then they also agree on \succ . Lemma 1 implies that

$$\phi(\succ) = (1 - \alpha)\phi(\succ_{-j}) + \alpha\pi_j = (1 - \beta)\phi(\succ_{-i}) + \beta\pi_i$$

for some $\alpha, \beta \in \mathbb{R} - \{1\}$. To make notation less cumbersome, we write $\sigma_k(\succ_{-j}) = \sigma_k^j$ and $\lambda_k(\succ_k) = \lambda_k$ for $k \in I$ and $J \subset I - \{i\}$ for the rest of this step. Four cases arise, depending on whether α and β are greater or smaller than 1.

Case 1. Assume $\alpha, \beta < 1$. By definition of σ , we have that $\sigma_k = \sigma_k^j$ for all $k \in I - \{j\}$ and $\sigma_k = \sigma_k^i$ for all $k \in I - \{i\}$. In particular, $\sigma_k^i = \sigma_k^j$ for $k \in I - \{i, j\}$. Moreover, either $\sigma_k^{ij} = \sigma_k^i$ for all $k \in I - \{i, j\}$ or $\sigma_k^{ij} = -\sigma_k^i$ for all k . Let $s = 1$ in the former case and $s = -1$ otherwise. Then,

$$\phi(\succ) \equiv s \underbrace{\left(\sum_{k \in I - \{i, j\}} \sigma_k^{ij} \lambda_k \pi_k \right)}_{\phi(\succ_{-i, j})} + \sigma_i^j \lambda_i \pi_i + \alpha' \pi_j = s \left(\sum_{k \in I - \{i, j\}} \sigma_k^{ij} \lambda_k \pi_k \right) + \beta' \pi_i + \sigma_j^i \lambda_j \pi_j$$

for some $\alpha', \beta' \in \mathbb{R}$. Affine independence implies that $\alpha' = \sigma_j^i \lambda_j = \sigma_j \lambda_j$. Moreover, $\sigma_i^j = \sigma_i$ and $s \sigma_k^{ij} = \sigma_k^i = \sigma_k$. So $\phi(\succ) = \sum_{k \in I} \sigma_k \lambda_k \pi_k$, which concludes this case.

Case 2. Assume $\alpha > 1$ and $\beta < 1$. By definition of σ , we have that $\sigma_k = -\sigma_k^j$ for all $k \in I - \{j\}$ and $\sigma_k = \sigma_k^i$ for all $k \in I - \{i\}$. In particular, $\sigma_k^i = -\sigma_k^j$ for $k \in I - \{i, j\}$. Moreover, either $\sigma_k^{ij} = \sigma_k^j$ for all $k \in I - \{i, j\}$ or $\sigma_k^{ij} = -\sigma_k^j$ for all k . Let $s = 1$ in the former case and $s = -1$ otherwise. Then,

$$\begin{aligned} \phi(\succ) &\equiv -\phi(\succ_{-j}) + \alpha' \pi_j \equiv -s \sum_{k \in I - \{i, j\}} \sigma_k^{ij} \lambda_k \pi_k - \sigma_i^j \lambda_i \pi_i + \alpha' \pi_j, \text{ and} \\ &\equiv \phi(\succ_{-i}) + \beta' \pi_i \equiv -s \sum_{k \in I - \{i, j\}} \sigma_k^{ij} \lambda_k \pi_k + \beta' \pi_i + \sigma_j^i \lambda_j \pi_j \end{aligned}$$

for some $\alpha', \beta' \in \mathbb{R}$. The second equality in the second line follows from $-s \sigma_k^{ij} = -\sigma_k^j = \sigma_k^i$ for $k \in I - \{i, j\}$. Affine independence implies that $\alpha' = \sigma_j^i \lambda_j = \sigma_j \lambda_j$. Moreover, $-\sigma_i^j = \sigma_i$ and $-s \sigma_k^{ij} = -\sigma_k^j = \sigma_k$. So $\phi(\succ) = \sum_{k \in I} \sigma_k \lambda_k \pi_k$.

The remaining two cases are analogous to the two we have examined and therefore omitted.

Step 7. We have shown that ϕ and $\bar{\phi}$ agree for groups $I = \{1, 2, l\}$ for all l , which we use for the base case $|I| = 3$. Let $I = \{1, i, j\}$ for distinct $i, j \in \mathbb{N} - \{1\}$ and $\succ \in \mathcal{R}^I$ such that π_1, π_i , and π_j are affinely independent. Observe that ϕ and $\bar{\phi}$ agree on the subprofiles \succ_{-i} and \succ_{-j} of \succ by Step 5. Since $\bar{\phi}(\succ_{-i, j}) = \pi_1, \pi_i$, and π_j are affinely independent, Step 6 implies that ϕ and $\bar{\phi}$ agree on \succ . With a second application of the same argument, we get that ϕ and $\bar{\phi}$ agree on all profiles of three agents with affinely independent beliefs.

In the rest of the proof, we deal with the case $|I| \geq 4$. Moreover, we assume that π has dimension at least 3 for now.

Case 1. Suppose π_{-k} has dimension 2 for some $k \in I$. Thus, all beliefs in π_{-k} are linear combinations of π_i and π_j for distinct but otherwise arbitrary $i, j \in I - \{k\}$. Since π has dimension 3, π_k is not in the affine hull of the beliefs in π_{-k} . So any subprofile of π with at least three agents one of which is k has dimension 3. By the induction hypothesis, ϕ and $\bar{\phi}$ agree on such profiles except for possibly π itself. In particular, they agree on \succsim_{-i} and \succsim_{-j} . Moreover, $\bar{\phi}(\succsim_{-i,j}) = \alpha\pi_i + \beta\pi_j + \gamma\pi_k$ for $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma \neq 0$ by Lemma 2. So $\bar{\phi}(\succsim_{-i,j}), \pi_i$, and π_j are affinely independent. By Step 6, we get that ϕ and $\bar{\phi}$ agree on \succsim .

Case 2. The remaining case is that π_{-k} has dimension 3 for all $k \in I$. The induction hypothesis implies that ϕ and $\bar{\phi}$ agree on \succsim_{-j} and \succsim_{-i} . The argument in the proof of Lemma 3 also applies to $\bar{\phi}$, and so we can choose distinct $i, j \in I$ such that $\bar{\phi}(\succsim_{-i,j}), \pi_j$, and π_i are affinely independent. We again conclude from Step 6 that ϕ and $\bar{\phi}$ agree on \succsim .

Now let \succsim be an arbitrary profile for agents in I . We have covered the case when π has dimension at least 3. If π has dimension 2, let $i \in \mathbb{N} - I$ and $(\tilde{\succsim})$ be a profile for agents in $I \cup \{i\}$ such that $\tilde{\succsim}_{-i} = \succsim$ and $\tilde{\pi}$ has dimension 3. Then $\phi(\tilde{\succsim}) = \bar{\phi}(\tilde{\succsim})$ and so Step 5 applies, which gives $\phi(\succsim) = \bar{\phi}(\succsim)$. Single-agent profiles are covered by Lemma 2. \square

A.2. Aggregation of Utility Functions

To begin with, it is useful to clarify the linear algebra on $\bar{\mathcal{U}}$. Elements of $\bar{\mathcal{U}}$ are normalized representatives of a class of utility functions, consisting of all its positive affine transformations. Thus, we say that $(u_i)_{i \in I}$ are linearly independent if their span does not include any utility function that is equivalent to the 0 element of $\bar{\mathcal{U}}$, that is, any constant utility function.

We show that the collective utility function of a group containing agent i is a linear combination of the utility function of i and that of the group without i . If the latter two utility functions are not equal to completely opposed, then i has positive weight in this linear combination. In Lemma 6, we leverage this fact to prove that the collective utility function of any group is a *positive* linear combination of the utility functions of its members.

Lemma 5. *Assume that Φ satisfies monotonicity. Let $I \in \mathcal{I}$, $i \in I$, and $\succsim \in \mathcal{R}^I$. Then $\psi(\succsim) = \alpha\psi(\succsim_{-i}) + \beta u_i$ for some $\alpha, \beta \in \mathbb{R}$. Moreover, if $u_i \neq \pm\psi(\succsim_{-i})$, then $\beta > 0$ and β is unique.*

Proof. Let $\succsim = \Phi(\succsim)$, $\succsim_{-i} = \Phi(\succsim_{-i})$, and $\pi_{-i} = \phi(\succsim_{-i})$. We start in the same way as for Lemma 1. Monotonicity implies that $f \sim g$ whenever $f \sim_{-i} g$, $f \sim_i g$, $\pi_{-i} \circ f^{-1} = \pi_i \circ f^{-1}$, and $\pi_{-i} \circ g^{-1} = \pi_i \circ g^{-1}$. Thus, it follows from Theorem 1 of Gilboa et al. (2004) that $\psi(\succsim) = \alpha\psi(\succsim_{-i}) + \beta u_i$ for some $\alpha, \beta \in \mathbb{R}$.

If $u_i \neq \pm\psi(\succsim_{-i})$, then β is unique. Moreover, we can find probability distributions p and q on X with finite support such that $\psi(\succsim_{-i})(p) = \psi(\succsim_{-i})(q)$ and $u_i(p) > u_i(q)$. Liapounoff's (1940) theorem allows us to construct acts f and g with the following properties: they induce the distributions p and q under π_{-i} and π_i , that is, $p = \pi_{-i} \circ f^{-1} = \pi_i \circ f^{-1}$ and $q = \pi_{-i} \circ g^{-1} = \pi_i \circ g^{-1}$; Thus, $f \sim_{-i} g$ and $f \succ_i g$. Since Φ is monotonic, we get that $f \succ g$. From Lemma 1, we know that $\phi(\succsim)$ is an affine combination of π_{-i} and π_i , and so $\phi(\succsim) \circ f^{-1} = p$ and $\phi(\succsim) \circ g^{-1} = q$. It follows that $f \succ g$ if and only if $\psi(\succsim)(p) > \psi(\succsim)(q)$. Thus, $\beta > 0$. \square

Lemma 6. *Assume that Φ satisfies monotonicity and faithfulness. Let $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$. Then $\psi(\succsim) = \sum_{i \in I} \mu_i u_i$ for some $\mu \in \mathbb{R}^I$. If $(u_i)_{i \in I}$ are linearly independent, $\mu \in \mathbb{R}_{++}^I$ and μ is unique.*

Proof. The first part is a straightforward corollary of Lemma 5. For the second part, assume that $(u_i)_{i \in I}$ are linearly independent. Let $\mu \in \mathbb{R}^I$ such that $\psi(\succsim) = \sum_{i \in I} \mu_i u_i$. Linear independence implies that μ is unique and $u_i \neq \pm \psi(\succsim_{-i})$ for all $i \in I$. Thus, Lemma 5 implies that $\psi(\succsim) = \alpha \psi(\succsim_{-i}) + \beta u_i$ for $\alpha \in \mathbb{R}$ and $\beta > 0$. Since $\psi(\succsim_{-i})$ is a linear combination of $(u_j)_{j \in I - \{i\}}$ and μ is unique, it follows that $\mu_i = \beta > 0$. \square

The dimension of a vector of utility functions $\mathbf{u} \in \mathcal{U}^I$ is the maximal number of linearly independent utility functions in $\{u_i : i \in I\}$. The next lemma is the analogue of Lemma 3. Its proof is similar and therefore omitted.

Lemma 7. *Assume that Φ satisfies restricted monotonicity. Let $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$ with $\psi(\succsim) \neq 0$. If \mathbf{u} has dimension at least 3, there are distinct $i, j \in I$ such that $\psi(\succsim_{-i,j}), u_i$, and u_j are linearly independent.*

In general, μ may depend on \succsim . The content of the next lemma is that μ_i must not depend on \succsim_{-i} . For the rest of this section, we assume that *beliefs and utility functions are pairwise distinct in any profile*.

Lemma 8. *Assume that $|X| \geq 4$ and Φ satisfies monotonicity and faithfulness. Then there is $\mu : \mathcal{R} \rightarrow \mathbb{R}_{++}^{\mathbb{N}}$ such that $\psi(\succsim) \equiv \sum_{i \in I} \mu_i(\succsim_i) u_i$ for all $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$.*

Proof. The first part is very similar to the construction of λ in the proof of Lemma 4. For $l \in \mathbb{N} - \{1, 2\}$, let $I_l = \{1, 2, l\}$ and \mathcal{R}_l be the set of all $\succsim \in \mathcal{R}^{I_l}$ such that u_1, u_2 , and u_l are linearly independent. Let $l \in \mathbb{N} - \{1, 2\}$ be arbitrary and fix some $\tilde{\succsim} \in \mathcal{R}_l$. By Lemma 6, there is a unique function $\nu : \mathcal{R}_l \rightarrow \mathbb{R}_{++}^{I_l}$ such that $\psi(\succsim) = \sum_{i \in I_l} \nu_i(\succsim) u_i$ for all $\succsim \in \mathcal{R}_l$. For $i, j \in I_l$ and $\succsim \in \mathcal{R}_l$, let $\lambda^{i,j}(\succsim) = \frac{\nu_j(\succsim_{-j, \tilde{\succsim}_j}) \nu_i(\succsim)}{\nu_j(\tilde{\succsim}) \nu_i(\succsim_{-j, \tilde{\succsim}_j})}$. The fact that ν maps to $\mathbb{R}_{++}^{I_l}$ ensures that $\lambda^{i,j}$ is well-defined and positive. Then, let $\mu_i(\succsim_i) = \frac{\nu_i(\tilde{\succsim})}{\lambda^{i,j}(\tilde{\succsim})}$. Note that the projection of \mathcal{R}_l to \mathcal{R} that returns the preferences of i is onto, and so μ_i is a function on all of \mathcal{R} . Here we use the assumption that $|X| \geq 4$, as otherwise \mathcal{R}_l is empty.

With the same arguments as in the proof of Lemma 4, we can show that $\frac{\nu_i(\tilde{\succsim})}{\nu_j(\tilde{\succsim})}$ is independent of $\tilde{\succsim}_k$ for $k \in I_l - \{i, j\}$, that $\lambda^{i,j}$ is independent of i and j , and that μ_i is well-defined. Then we have

$$\psi(\succsim) = \sum_{i \in I_l} \nu_i(\succsim) u_i \equiv \sum_{i \in I} \frac{\nu_i(\tilde{\succsim})}{\lambda^{i,j}(\tilde{\succsim})} u_i = \sum_{i \in I} \mu_i(\succsim_i) u_i,$$

where $j_i \in I - \{i\}$ for all $i \in I$.

Since l was arbitrary, we have now defined μ_i for each $i \in \mathbb{N}$. However, we have defined μ_1 and μ_2 multiple times, once for each $l \in \mathbb{N} - \{1, 2\}$. So we have to check that these definitions are not conflicting. It follows from Lemma 1 that the ratio between μ_1 and μ_2 is the same for each triple $\{1, 2, l\}$. Thus, we can define μ_1 and μ_2 as obtained for, say, $l = 3$ and scale the triples (μ_1, μ_2, μ_l) obtained for the remaining l appropriately.

μ defines a function that returns a collective utility function for every profile. For $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$, let $\bar{\psi}(\succsim) \equiv \sum_{i \in I} \mu_i(\succsim_i) u_i$. The following two observations will carry us a long way in the rest of the proof.

Step 1. Let \succsim be a profile for agents in I such that \mathbf{u} has dimension 3 and \mathbf{u}_{-i} has dimension 2. If ψ and $\bar{\psi}$ agree on \succsim , then they also agree on \succsim_{-i} . By Lemma 5 and the assumption, we have

$$\psi(\succsim) = \alpha \psi(\succsim_{-i}) + \beta u_i \equiv \sum_{j \in I - \{i\}} \mu_j(\succsim_j) u_j + \mu_i(\succsim_i) u_i$$

for some $\alpha, \beta \in \mathbb{R}$. Since u_i is not in the span of $(u_j)_{j \in I - \{i\}}$, we get that $\psi(\succsim_{-i}) \equiv \sum_{j \in I - \{i\}} \mu_j(\succsim_j) u_j \equiv \bar{\psi}(\succsim_{-i})$.

Step 2. Let \succsim be a profile for agents in I and $i, j \in I$. If $\bar{\psi}(\succsim_{-i,j}), u_i$, and u_j are linearly independent and ψ and $\bar{\psi}$ agree on \succsim_{-i} and \succsim_{-j} , then they also agree on \succsim .

Lemma 5 and Lemma 6 imply that

$$\begin{aligned} \psi(\succsim) &\equiv \psi(\succsim_{-j}) + \alpha u_j \equiv \overbrace{\sum_{k \in I - \{i,j\}} \mu_k(\succsim_k) u_k + \mu_i(\succsim_i) u_i}^{\equiv \bar{\psi}(\succsim_{-j})} + \alpha' u_j, \text{ and} \\ \psi(\succsim) &\equiv \psi(\succsim_{-i}) + \beta u_i \equiv \underbrace{\sum_{k \in I - \{i,j\}} \mu_k(\succsim_k) u_k}_{\equiv \bar{\psi}(\succsim_{-i,j})} + \beta' u_i + \mu_j(\succsim_j) u_j \end{aligned}$$

for some $\alpha, \alpha', \beta, \beta' \in \mathbb{R}_{++}$. Linear independence of $\bar{\psi}(\succsim_{-i,j}), u_i$, and u_j implies that $\alpha' = \mu_j(\succsim_j)$, and so ψ and $\bar{\psi}$ agree on \succsim .

With all this in place, we can finish the proof. First we show by induction over $|I|$ that ψ and $\bar{\psi}$ agree on all profiles where \mathbf{u} has dimension at least 3. Later we take care of the remaining profiles later.

The base case is $|I| = 3$. Let $\succsim \in \mathcal{R}^I$. First assume that $I = \{1, i, j\}$ for distinct $i, j \in \mathbb{N} - \{1\}$. We have shown that ψ and $\bar{\psi}$ agree for groups of the form $\{1, 2, l\}$ for any l . Thus Step 1 implies that they agree on \succsim_{-i} and \succsim_{-j} . Moreover, $\bar{\psi}(\succsim_{-i,j}) = u_1, u_i$, and u_j are linearly independent. So Step 2 implies that ψ and $\bar{\psi}$ agree on \succsim . A second iteration of the same argument implies that they agree on profiles for three arbitrary agents.

Now we deal with the case $|I| \geq 4$ and again assume that \mathbf{u} has dimension at least 3.

Case 1. Suppose $\psi(\succsim) = 0$ or $\bar{\psi}(\succsim) = 0$. We show that $\psi(\succsim) = 0$ if and only if $\bar{\psi}(\succsim) = 0$. Assume for contradiction that $\psi(\succsim) \neq 0$ and $\bar{\psi}(\succsim) = 0$. By Lemma 7, we can find distinct i, j such that $\psi(\succsim_{-i,j}), u_i$, and u_j are linearly independent. Since $|I| \geq 4$, $\mathbf{u}_{-i,j}$ has dimension at least 2. If it has dimension exactly 2, then either \mathbf{u}_{-i} or \mathbf{u}_{-j} has dimension 3, as otherwise \mathbf{u} would have dimension 2. Suppose \mathbf{u}_{-j} has dimension at least 3. By the induction hypothesis, $\psi(\succsim_{-j}) = \bar{\psi}(\succsim_{-j})$. Since $\psi(\succsim_{-i,j}), u_i$, and u_j are linearly independent, we can conclude that $u_j \neq \pm \psi(\succsim_{-j})$. But $\bar{\psi}(\succsim) \equiv \bar{\psi}(\succsim_{-j}) + \alpha u_j$ for some $\alpha > 0$, and so since $\pm u_j \neq \bar{\psi}(\succsim_{-j}) = \bar{\psi}(\succsim_{-j})$, we get $\bar{\psi}(\succsim) \neq 0$.

The proof is similar if $\psi(\succ) = 0$ and $\bar{\psi}(\succ) \neq 0$. Note however, that we find i, j such that $\bar{\psi}(\succ_{-i,j}), u_i$, and u_j are linearly independent not directly by Lemma 7, but by the same argument as in its proof.

Case 2. Suppose \mathbf{u}_{-k} has dimension 2 for some $k \in I$. Thus, all beliefs in \mathbf{u}_{-k} are linear combinations of u_i and u_j for distinct but otherwise arbitrary $i, j \in I - \{k\}$. Since \mathbf{u} has dimension 3, u_k is not in the span of the utility functions in \mathbf{u}_{-k} . So any subprofile of \mathbf{u} with at least 3 agents one of which is k has dimension 3. By the induction hypothesis, ψ and $\bar{\psi}$ agree on such profiles except for possibly \succ itself. In particular, they agree on \succ_{-i} and \succ_{-j} . Moreover, $\bar{\psi}(\succ_{-i,j}) = \alpha u_i + \beta u_j + \gamma u_k$ for $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma > 0$ by definition of $\bar{\psi}$. So $\bar{\psi}(\succ_{-i,j}), u_i$, and u_j are linearly independent. Step 2 implies that $\psi(\succ) = \bar{\psi}(\succ)$.

Case 3. The remaining case is that \mathbf{u}_{-k} has dimension at least 3 for all $k \in I$. The induction hypothesis implies that ψ and $\bar{\psi}$ agree on \succ_{-i} and \succ_{-j} . By Lemma 7 we can choose distinct $i, j \in I$ such that $\psi(\succ_{-i,j}), u_i$, and u_j are linearly independent. If $\mathbf{u}_{-i,j}$ has dimension 3, the induction hypothesis implies that ψ and $\bar{\psi}$ agree on $\succ_{-i,j}$. If $\mathbf{u}_{-i,j}$ has dimension 2, then we use the fact that \mathbf{u}_{-i} has dimension 3 to apply Step 1 and conclude that ψ and $\bar{\psi}$ agree on $\succ_{-i,j}$. In either case, $\bar{\psi}(\succ_{-i,j}), u_i$, and u_j are linearly independent. So Step 2 implies that ψ and $\bar{\psi}$ agree on \succ .

Now let \succ be an arbitrary profile for agents in I . We have covered the case when \mathbf{u} has dimension at least 3. If \mathbf{u} has dimension 2, let $i \in \mathbb{N} - I$ and $\tilde{\succ}$ be a profile for agents in $I \cup \{i\}$ such that $\tilde{\succ}_{-i} = \succ$ and $\tilde{\mathbf{u}}$ has dimension 3. Then $\psi(\tilde{\succ}) = \bar{\psi}(\tilde{\succ})$, and so Step 1 implies $\psi(\succ) = \bar{\psi}(\succ)$. Single-agent profiles are covered by Lemma 6. \square

B. Implications of Continuity

Let us first define topologies on Π and \mathcal{U}^* , which we do in the same way as for $\bar{\mathcal{R}}$. For $\pi, \pi' \in \Pi$, the uniform metric $\sup\{|\pi(E) - \pi'(E)| : E \in \Sigma\}$ gives a topology on Π . For $u, u' \in \mathcal{U}$ (note the absence of constant utility function 0), we also use the uniform metric $\sup\{|u(x) - u'(x)| : x \in X\}$. The topology on $\bar{\mathcal{U}}$ is that of \mathcal{U} plus the entire set $\bar{\mathcal{U}}$. So the only neighborhood of 0 is the set $\bar{\mathcal{U}}$ itself. This is the topology $\bar{\mathcal{U}}$ inherits from the space of all utility functions equipped with the uniform metric when forming the quotient via normalization.

The mappings from preference relations to beliefs and utility functions are now continuous. Likewise, the inverse operation mapping a pair of belief and utility function to a preference relation is continuous. Lemma 5 of Dietrich (2019) is the equivalent of this statement in his framework. To ease notation in the proof of the next lemma, when E is an event and x, y are consequences, we write xEy for the act which yields x for states in E and y for states in $\Omega - E$.

Lemma 9. *The correspondence $\pi(\succ)$ and the function $u(\succ)$ mapping $\succ \in \bar{\mathcal{R}}$ to the beliefs and the utility function representing \succ are (upper-hemi) continuous. Moreover, the function $\succ(\pi, u)$ mapping each pair of belief and utility function to the preference relation it induces is continuous.*

Proof. Let (\succsim^n) be a sequence that converges to \succsim in $\bar{\mathcal{R}}$. For each n , let $\pi^n \in \pi(\succsim^n)$ and $u^n = u(\succsim^n)$.

First we show that (u^n) converges to $u = u(\succsim)$. Let $x \in X$ and f_x be the act that returns x in all states. We have $\sup\{|u^n(x) - u(x)| : x \in X\} = \sup\{|\mathbb{E}_{\succsim^n}(f_x) - \mathbb{E}_{\succsim}(f_x)| : x \in X\}$, and so (u^n) converges uniformly to u .

Second, assume that (π^n) converges to $\pi' \in \Pi$. We have to show $\pi' \in \pi(\succsim)$. If $u = 0$, then $\pi(\succsim) = \Pi$ and there is nothing to show. Otherwise, $\pi(\succsim) = \{\pi\}$ for some $\pi \in \Pi$. We show that (π^n) converges to π , which implies $\pi = \pi'$. Since $u \neq 0$, we can choose $x, y \in X$ such that $u(x) > u(y)$. Then, for large enough n , $\sup\{|\pi^n(E) - \pi(E)| : E \in \Sigma\} = \sup\{|\frac{\mathbb{E}_{\succsim^n}(xEy) - u^n(y)}{u^n(x) - u^n(y)} - \frac{\mathbb{E}_{\succsim}(xEy) - u(y)}{u(x) - u(y)}| : E \in \Sigma\} \leq \frac{2}{u(x) - u(y)} \sup\{|\mathbb{E}_{\succsim^n}(xEy) - \mathbb{E}_{\succsim}(xEy)| : E \in \Sigma\}$, and so (π^n) converges uniformly to π .

Conversely, assume that (π^n) and (u^n) converge to π and u , respectively, and let $\succsim^n = \succsim(\pi^n, u^n)$ and $\succsim = \succsim(\pi, u)$ be the induced preference relations. For $\epsilon > 0$, let $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $\sup\{|\pi^n(E) - \pi(E)| : E \in \Sigma\} < \frac{\epsilon}{2}$ and $\sup\{|u^n(x) - u(x)| : x \in X\} < \frac{\epsilon}{2}$. Then, for all $n \geq n_0$ and $f \in \mathcal{F}$,

$$\begin{aligned} |\mathbb{E}_{\succsim^n}(f) - \mathbb{E}_{\succsim}(f)| &= \left| \int_{\Omega} (u^n \circ f) d\pi^n - \int_{\Omega} (u \circ f) d\pi \right| \\ &\leq \left| \int_{\Omega} (u^n \circ f - u \circ f) d\pi^n \right| + \left| \int_{\Omega} (u \circ f) d\pi^n - \int_{\Omega} (u \circ f) d\pi \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus \mathbb{E}_{\succsim^n} converges uniformly to \mathbb{E}_{\succsim} . \square

What still separates us from the desired statement of Proposition 1 after establishing Lemma 4 and Lemma 8 is that Lemma 4 allows negative weights for beliefs and that both lemmas only apply if beliefs and utility functions are pairwise distinct. The latter is not hard to deal with, but showing that weights cannot be negative takes more work.

Proposition 1. *Let $|X| \geq 4$ and Φ be an aggregation function. Then the following are equivalent.*

- (i) Φ satisfies restricted monotonicity, faithfulness, no belief imposition, and continuity
- (ii) There are continuous functions $\lambda, \mu : \mathcal{R} \rightarrow \mathbb{R}_{++}^{\mathbb{N}}$ such that for all $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$, $\Phi(\succsim)$ is represented by $\frac{1}{\sum_{i \in I} \lambda_i(\succsim_i)} \sum_{i \in I} \lambda_i(\succsim_i) \pi_i$ and $\sum_{i \in I} \mu_i(\succsim_i) u_i$

Proof. One can check easily that (ii) implies (i). The rest of the proof will establish that (i) implies (ii).

Let λ, μ , and for all $I \in \mathcal{I}$, σ^I be the functions obtained from Lemma 4 and Lemma 8.

Step 1. We show that μ is continuous. Let $i, j \in \mathbb{N}$ and $\succsim \in \mathcal{R}^{\{i, j\}}$ such that $\pi_i \neq \pi_j$ and $u_i \neq \pm u_j$; let (\succsim_i^n) be a sequence in \mathcal{R} converging to \succsim_i and $\succsim^n = (\succsim_i^n, \succsim_j)$. By assumption, Φ is continuous, and by Lemma 9, the mapping from preference relations to the corresponding utility functions is continuous. Thus, $u^n = \psi(\succsim^n) \equiv \mu_i(u_i^n)u_i^n + \mu_j(u_j)u_j$ converges to $u = \psi(\succsim) \equiv \mu_i(u_i)u_i + \mu_j(u_j)u_j$. First, $\mu_i(\succsim_i^n)$ is bounded, as otherwise, a subsequence of (u^n)

would converge to u_i . But this is impossible, since $\mu_j(u_j) \neq 0$ and $u_i \neq \pm u_j$. Now if α is an accumulation point of $(\mu_i(u_i^n))$, then $\alpha u_i + \mu_j(u_j)u_j \equiv u$, since (u^n) converges to u . But $\alpha u_i + \mu_j(u_j)u_j \equiv \beta u_i + \mu_j(u_j)u_j$ if and only if $\alpha = \beta$. So $(\mu_i(u_i^n))$ is bounded and has a unique accumulation point. Thus, it converges to $\mu_i(u_i)$.

Step 2. Let $i, j \in \mathbb{N}$ and $\succ \in \mathcal{R}^{\{i,j\}}$ such that $\pi_i \neq \pi_j$, $u_i \neq \pm u_j$, and $\Phi(\succ)$ is not complete indifference. We show that $\sigma_i^{\{i,j\}} \lambda_i$ is continuous at \succ . (For convenience, we will omit the superscript $\{i, j\}$ from now on.) Let (\succ^n) be a sequence of profiles converging to \succ . Let $\alpha^n = \sigma_i(\succ^n) \lambda_i(\succ_i^n)$ and $\beta^n = \sigma_j(\succ^n) \lambda_j(\succ_j^n)$, and $\alpha = \sigma_i(\succ) \lambda_i(\succ_i)$ and $\beta = \sigma_j(\succ) \lambda_j(\succ_j)$.

First we prove convergence when j 's preferences remain constant at \succ_j . Let $\bar{\succ}^n = (\succ_i^n, \succ_j)$, $\bar{\alpha}^n = \sigma_i(\bar{\succ}^n) \lambda_i(\succ_i^n)$, and $\bar{\beta}^n = \sigma_j(\bar{\succ}^n) \lambda_j(\succ_j)$. We need to show that $(\bar{\alpha}^n)$ converges to α . Note that $\bar{\beta}^n$ can only vary in sign but not in absolute value. Since Φ and the correspondence mapping preference relations to the corresponding beliefs are continuous, we have that $\bar{\alpha}^n \pi_i^n + \bar{\beta}^n \pi_j \equiv \phi(\bar{\succ}^n) \rightarrow \phi(\succ) \equiv \alpha \pi_i + \beta \pi_j$. With the same reasoning as in Step 1, we get that $(\bar{\alpha}^n)$ is bounded and has a unique accumulation point. Thus, it converges to α . Similarly, $\sigma_j(\succ_i, \succ_j^n) \lambda_j(\succ_j^n)$ converges to β .

Now we show that (α^n) converges α . We already know that the sequences of absolute values of (α^n) and (β^n) converge to α and β , respectively. So any subsequence $(\alpha^{n_k}, \beta^{n_k})$ such that all α^{n_k} and all β^{n_k} have the same sign converges. By the same reasoning as in the previous paragraph, we conclude that $\alpha^{n_k} \pi_i^{n_k} + \beta^{n_k} \pi_j^{n_k} \equiv \phi(\succ^{n_k}) \rightarrow \phi(\succ) \equiv \alpha \pi_i + \beta \pi_j$. Since $\pi_i \neq \pi_j$, this implies that $(\alpha^{n_k}, \beta^{n_k})$ converges to (α, β) . Thus (α^n, β^n) converges to (α, β) .

Step 3. Now we deduce that σ_i is always equal to 1. Assume for contradiction that there is a profile $\tilde{\succ} \in \mathcal{R}^{\{i,j\}}$ such that $\tilde{\pi}_i \neq \tilde{\pi}_j$, $\tilde{u}_i \neq \pm \tilde{u}_j$, $\Phi(\tilde{\succ})$ is not complete indifference, and $\sigma_i(\tilde{\succ}) = -1$. Since $\sigma_i \lambda_i$ and $\sigma_j \lambda_j$ are continuous at $\tilde{\succ}$ by Step 2, we can find a neighborhood of $\tilde{\succ}$ such that $\sigma_i(\succ) = -1$ for all profiles \succ contained in it. In particular, we can find $\epsilon > 0$ such that $\sigma_i(\succ) = -1$ whenever $\succ_i = \tilde{\succ}_i$, $u_j = \tilde{u}_j$, and $\sup\{|\pi_j(E) - \tilde{\pi}_j(E)| : E \in \Sigma\} < \epsilon$. Let $\tilde{\mathcal{P}}$ be this set of profiles. By Liapounoff's (1940) theorem, we can find an event E such that $\tilde{\pi}_i(E) = \tilde{\pi}_j(E) = \frac{\epsilon}{2}$. But then $\tilde{\mathcal{P}}$ contains a profile \succ such that $\sigma_i(\succ) = -1$, $\pi_i(E) = \tilde{\pi}_i(E) = \frac{\epsilon}{2}$ and $\pi_j(E) = 0$. This is not possible, since $\sigma_i(\succ) \lambda_i(\succ_i) \pi_i(E) + \sigma_j(\succ) \lambda_j(\succ_j) \pi_j(E)$ would be negative.

Since j was arbitrary and σ^I satisfies the restriction on the ratio of σ_i^I and σ_j^I stated in Lemma 4, it follows that σ_i^I is constant at 1 for all I . Since we have shown that $\sigma_i \lambda_i$ is continuous, so is λ_i .

(Alternatively, one could show that the set of profiles where beliefs and utility functions are pairwise distinct and Φ is not complete indifference is connected. Then the fact that $\sigma_i \lambda_i$ is continuous and never 0 implies that it cannot change sign.)

Step 4. Let $\bar{\Phi}$ be the aggregation function where $\bar{\Phi}(\succ)$ is represented by $\bar{\phi}(\succ) \equiv \sum_{i \in I} \lambda_i(\succ_i) \pi_i$ and $\bar{\psi}(\succ) \equiv \sum_{i \in I} \mu_i(\succ_i) u_i$ for every profile \succ with agents in $I \in \mathcal{I}$. We know that Φ and $\bar{\Phi}$ agree on all profiles for which beliefs and utility functions are pairwise distinct. These profiles are dense in \mathcal{R}^I for all I . Our task is to show that they agree on an arbitrary profile $\succ \in \mathcal{R}^I$.

Case 1. Suppose neither $\Phi(\succ)$ nor $\bar{\Phi}(\succ)$ is complete indifference. By Lemma 9, Step 1, and

Step 2, $\phi, \psi, \bar{\phi}$, and $\bar{\psi}$ are continuous at \succcurlyeq . Moreover, the pairs ϕ and $\bar{\phi}$ and ψ and $\bar{\psi}$ agree on a set of profiles with \succcurlyeq in its closure. Thus, $\phi(\succcurlyeq) = \bar{\phi}(\succcurlyeq)$ and $\psi(\succcurlyeq) = \bar{\psi}(\succcurlyeq)$. It follows that $\Phi(\succcurlyeq) = \bar{\Phi}(\succcurlyeq)$.

Case 2. Suppose that $\Phi(\succcurlyeq)$ is complete indifference. (The proof is analogous if $\bar{\Phi}(\succcurlyeq)$ is complete indifference.) Let $i \in \mathbb{N} - I$ with preferences \succcurlyeq_i such that $u_i \neq \pm \bar{\psi}(\succcurlyeq)$. Then for $\succcurlyeq_{+i} = (\succcurlyeq, \succcurlyeq_i)$, by Lemma 5, $\psi(\succcurlyeq_{+i}) = u_i$ and $\bar{\psi}(\succcurlyeq_{+i}) \equiv \bar{\psi}(\succcurlyeq) + \alpha u_i$ for some $\alpha > 0$. In particular, $\psi(\succcurlyeq_{+i}), \bar{\psi}(\succcurlyeq_{+i}) \neq 0$. Thus Case 1 implies that $\psi(\succcurlyeq_{+i}) = \bar{\psi}(\succcurlyeq_{+i})$. Since $u_i \neq \pm \bar{\psi}(\succcurlyeq)$, this can only be if $\bar{\psi}(\succcurlyeq) = 0$, and hence $\bar{\Phi}(\succcurlyeq)$ is complete indifference. □

C. Implications of Independence of Redundant Acts

Using independence of redundant acts, we derive a lemma which, together with Proposition 1, concludes the proof of Theorem 1. But first we need an auxiliary statement. Recall that a function is simple if it has finite range.

Lemma 10. *Let $I \in \mathcal{I}$ and $i \in I$; let $\succcurlyeq \in \mathcal{R}^I$ such that u_j is simple for $j \in I - \{i\}$. Then for every act f , there is a simple act g such that $f \sim_j g$ for all $j \in I$.*

Proof. Put differently, we want to show that for every act f , there is a simple act g such that $(\mathbb{E}_{\succcurlyeq_j}(f))_{j \in I} = (\mathbb{E}_{\succcurlyeq_j}(g))_{j \in I}$.

We first show that the sets $X^+ = \{x \in X : u_i(x) \geq \mathbb{E}_{\succcurlyeq_i}(f)\}$ and $X^- = \{x \in X : u_i(x) \leq \mathbb{E}_{\succcurlyeq_i}(f)\}$ are non-empty. If X^+ is empty, then $\Omega = \bigcup_{k \in \mathbb{N}} \{s \in \Omega : u_i(f(s)) \leq \mathbb{E}_{\succcurlyeq_i}(f) - \frac{1}{k}\}$. Note that all sets in this union are measurable. Since π_i is countably additive, there is k_0 such that $\pi_i(\{s \in \Omega : u_i(f(s)) \leq \mathbb{E}_{\succcurlyeq_i}(f) - \frac{1}{k_0}\}) = \epsilon > 0$. The fact that X^+ is empty then gives $\mathbb{E}_{\succcurlyeq_i}(f) \leq \mathbb{E}_{\succcurlyeq_i}(f) - \frac{\epsilon}{k_0}$, which is a contradiction. Similarly, one shows that X^- is non-empty.

Now let $V = \{\mathbf{u}_{-i}(x) : x \in X\} \subset \mathbb{R}^{I-\{i\}}$ be the range of \mathbf{u}_{-i} . Since all u_j are simple, V is finite. We partition the set of consequences X into the measurable sets $X_v = \mathbf{u}_{-i}^{-1}(v)$ and the set of states Ω into the measurable sets $E_v = f^{-1}(X_v)$ with v ranging over V . To define g , consider two cases. If $\pi_i(E_v) = 0$, choose $x_v \in X_v$ arbitrarily and let $g(s) = x_v$ for all $s \in E_v$. If $\pi_i(E_v) > 0$, then $\frac{1}{\pi_i(E_v)}\pi_i|_{E_v}$ is a probability measure on E_v , where $\pi_i|_{E_v}$ is π_i restricted to events contained in E_v . By the previous paragraph, the sets $X_v^+ = \{x \in X_v : u_i(x) \geq \frac{1}{\pi_i(E_v)} \int_{E_v} (u_i \circ f) d(\pi_i|_{E_v})\}$ and $X_v^- = \{x \in X_v : u_i(x) \leq \frac{1}{\pi_i(E_v)} \int_{E_v} (u_i \circ f) d(\pi_i|_{E_v})\}$ are non-empty. Thus, there are $x^+ \in X_v^+$ and $x^- \in X_v^-$ and $\alpha \in [0, 1]$ such that $\alpha u_i(x^+) + (1 - \alpha)u_i(x^-) = \frac{1}{\pi_i(E_v)} \int_{E_v} (u_i \circ f) d(\pi_i|_{E_v})$. Since π_i is non-atomic, there is $E_v^+ \subset E_v$ such that $\pi_i(E_v^+) = \alpha \pi_i(E_v)$. We define $g(s) = x^+$ for $s \in E_v^+$ and $g(s) = x^-$ for $s \in E_v - E_v^+$. This gives

$$\begin{aligned} \int_{E_v} (u_i \circ f) d(\pi_i|_{E_v}) &= \pi_i(E_v)(\alpha u_i(x^+) + (1 - \alpha)u_i(x^-)) \\ &= \pi_i(E_v^+)u_i(x^+) + \pi_i(E_v - E_v^+)u_i(x^-) = \int_{E_v} (u_i \circ g) d(\pi_i|_{E_v}) \end{aligned}$$

Also, since \mathbf{u}_{-i} is constant on X_v , $\int_{E_v} (u_j \circ f) d(\pi_j|_{E_v}) = \int_{E_v} (u_j \circ g) d(\pi_j|_{E_v})$ for all $j \in I - \{i\}$. In summary, we have $(\mathbb{E}_{\succcurlyeq_j}(f))_{j \in I} = (\mathbb{E}_{\succcurlyeq_j}(g))_{j \in I}$. □

Lemma 11. *Let $\lambda, \mu: \mathcal{R} \rightarrow \mathcal{R}_{++}^{\mathbb{N}}$ be continuous functions; let Φ be an aggregation function such that for every $I \in \mathcal{I}$ and $\succsim \in \mathcal{R}^I$, $\Phi(\succsim)$ is represented by $\frac{1}{\sum_{i \in I} \lambda_i(\succsim_i)} \sum_{i \in I} \lambda_i(\succsim_i) \pi_i$ and $\sum_{i \in I} \mu_i(\succsim_i) u_i$. Then if Φ satisfies independence of redundant acts, λ and μ are constant.*

Proof. Let $i, j \in \mathbb{N}$ and $\succsim_i, \succsim_i' \in \mathcal{R}$. We want to show that $\lambda_i(\succsim_i) = \lambda_i(\succsim_i')$ and $\mu_i(\succsim_i) = \mu_i(\succsim_i')$. In the first step, we show that λ_i and μ_i are independent of π_i . In the rest of the proof, we show that they are independent of u_i , too.

Step 1. Assume that $u_i = u_i'$. First we show $\lambda_i(\succsim_i) = \lambda_i(\succsim_i')$. Let $\Lambda = \{E \in \Sigma: \pi_i(E) = \pi_i'(E)\}$. We construct a suitable belief for agent j . Let $E \in \Lambda$ such that $\pi_i(E) = \frac{1}{2}$. If $\pi_i = \pi_i'$, there is nothing to show. Otherwise, either $\pi_i|_E \neq \pi_i'|_E$ or $\pi_i|_{\Omega-E} \neq \pi_i'|_{\Omega-E}$. Assume the former is true. Then define π_j so that $\pi_j(F) = 2\pi_i(F)$ for every $F \subset E$ and $\pi_j(\Omega - E) = 0$. Moreover, choose $u_j \in \mathcal{U}$ so that u_j is simple and $u_j \neq \pm u_i$ and let \succsim_j be represented by π_j and u_j .

The set of acts to which we will apply independence of redundant acts is $\mathcal{G} = \{g \in \mathcal{F}: g \text{ is simple and } g^{-1}(x) \in \Lambda \text{ for every } x \in X\}$. To meet the antecedent of independence of redundant acts, we have to show that for every $f \in \mathcal{F}$, there is $g \in \mathcal{G}$ such that $f \sim_i g$ and $f \sim_j g$. (The choice of \mathcal{G} and $u_i = u_i'$ ensure that also $f \sim_i' g$.)

By Lemma 10, we may assume that f is simple. Define g as follows: let $f(\Omega) = \{x_1, \dots, x_k\}$ be the range of f . For every x_l , let $\alpha_l = \pi_i(E \cap f^{-1}(x_l))$ and $\alpha_l^c = \pi_i((\Omega - E) \cap f^{-1}(x_l))$. (Note that $\pi_j(E \cap f^{-1}(x_l)) = 2\alpha_l$.) Liapounoff's theorem allows us to find events $E_l \subset E$ and $E_l^c \subset \Omega - E$ in Λ such that $\pi_i(E_l) = \alpha_l$ and $\pi_i(E_l^c) = \alpha_l^c$. In fact, we can partition E and $\Omega - E$ into $\{E_1, \dots, E_k\}$ and $\{E_1^c, \dots, E_k^c\}$, respectively. Then let $g(s) = x_l$ for $s \in E_l \cup E_l^c$. One can check that $\pi_j(E_l \cup E_l^c) = 2\pi_i(E_l) = 2\alpha_l$. Thus, $\mathbb{E}_{\succsim_i}(f) = \mathbb{E}_{\succsim_i}(g)$ and $\mathbb{E}_{\succsim_j}(f) = \mathbb{E}_{\succsim_j}(g)$ and so $f \sim_i g$ and $f \sim_j g$.

Let $\succsim = f(\succsim_i, \succsim_j)$ and $\succsim' = f(\succsim_i', \succsim_j)$ and π, u, π', u' be the corresponding beliefs and utility functions. Independence of redundant acts applied to the profiles (\succsim_i, \succsim_j) and $(\succsim_i', \succsim_j)$ gives $g \succsim g'$ if and only if $g \succsim' g'$ for all $g, g' \in \mathcal{G}$.

Assume that $\lambda_i(\succsim_i) \neq \lambda_i(\succsim_i')$. First, since $u_j \neq \pm u_i$ and $u \equiv \lambda_i(u_i)u_i + \lambda_j(u_j)u_j$, \succsim cannot be complete indifference, and so we can find consequences x and y such that $x \succ y$. Recall that $\pi_i(E) = \pi_i'(E) = \frac{1}{2}$ and $\pi_j(E) = 1$. It follows that $\pi(E) \neq \pi'(E)$ and $\pi(E), \pi'(E) > \frac{1}{2}$. So there is an event $E' \subset E$ such that $E' \in \Lambda$, $\pi(E') \neq \frac{1}{2}$, and $\pi'(E') = \frac{1}{2}$. Thus, $x E' y$ and $y E' x$ are acts in \mathcal{G} but $x E' y \not\sim y E' x$ and $x E' y \sim' y E' x$. This contradicts independence of redundant acts and so $\lambda_i(\succsim_i) = \lambda_i(\succsim_i')$.

Second, assume that $\mu_i(\succsim_i) \neq \mu_i(\succsim_i')$. Since $u_j \neq \pm u_i$, it follows that $u \neq u'$ and we can find simple lotteries p and q on X such that $u(p) > u(q)$ but $u'(q) \geq u'(p)$. Since $\lambda_i(\succsim_i) = \lambda_i(\succsim_i')$ by the previous paragraph, $\pi(F) = \pi'(F)$ if and only if $F \in \Lambda$. So by Liapounoff's theorem, we can find acts g and g' in \mathcal{G} with $g \circ \pi = p$ and $g' \circ \pi' = q$. This gives $g \succ g'$ but $g' \succsim' g$, which contradicts independence of redundant acts. We conclude that $\mu_i(\succsim_i) = \mu_i(\succsim_i')$.

Step 2. By Step 1, we can view λ_i and μ_i as functions $\lambda_i(u_i)$ and $\mu_i(u_i)$ of u_i . We show that both these functions are constant.

Recall that \mathcal{U} consists of utility functions which are normalized to the unit interval, that is, $\inf_x u(x) = 0$ and $\sup_x u(x) = 1$. Let $\mathcal{U}' = \{u \in \mathcal{U}: \text{there exist } x, y \in X \text{ with } u(x) =$

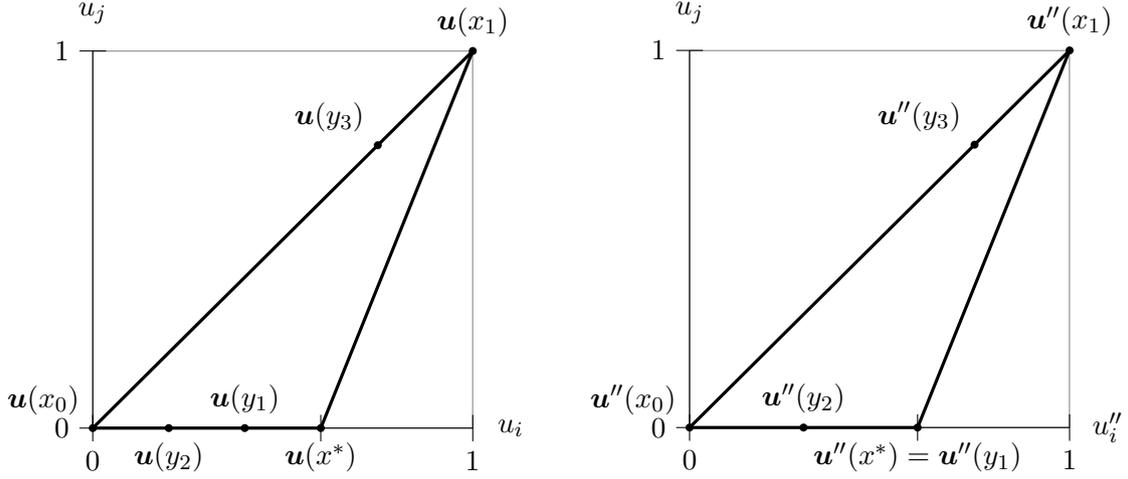


Figure 1: The images of $\mathbf{u} = (u_i, u_j)$ and $\mathbf{u}'' = (u''_i, u_j)$ in utility space. For example, $\mathbf{u}(x_0) = (u_i(x_0), u_j(x_0)) = (0, 0)$. The consequences y_1, y_2 , and y_3 are examples for the three cases in the definition of u''_i .

0 and $u(y) = 1$ be those utility functions for which the infimum and the supremum are attained. Observe that the closure of \mathcal{U}' is \mathcal{U} . Thus, since λ_i and μ_i are continuous, it suffices to show that they are constant on \mathcal{U}' . This we do now.

Let $u_i \in \mathcal{U}'$; let $x_0, x_1 \in X$ such that $u_i(x_0) = 0$ and $u_i(x_1) = 1$ and $x^* \in X - \{x_0, x_1\}$ be arbitrary; let u'_i be such that $u'_i(x) = u_i(x)$ for $x \in \{x_0, x_1, x^*\}$. We show that $\lambda_i(u_i) = \lambda_i(u'_i)$ and $\mu_i(u_i) = \mu_i(u'_i)$. Since $|X| \geq 4$, repeated application of this statement gives the same conclusion for all $u'_i \in \mathcal{U}'$.

Let u''_i be such that

$$u''_i(x) = \begin{cases} u_i(x^*) & \text{if } u_i(x) < u_i(x^*) \text{ and } u'_i(x) > u_i(x^*) \\ u'_i(x) & \text{if } u_i(x) < u_i(x^*) \text{ and } u'_i(x) \leq u_i(x^*) \\ u_i(x) & \text{if } u_i(x) \geq u_i(x^*) \end{cases}$$

Note that $u''_i(x) = u'_i(x) = u_i(x)$ for $x \in \{x_0, x_1, x^*\}$. We want to apply independence of redundant acts to profiles with utility functions (u_i, u_j) and (u''_i, u_j) and the set of acts $\mathcal{G} = \{f \in \mathcal{F} : f(\Omega) \subset \{x_0, x_1, x^*\}\}$. This requires choosing u_j appropriately. Let u_j be such that

$$u_j(x) = \begin{cases} 0 & \text{if } u_i(x) \leq u_i(x^*) \\ u_i(x) & \text{otherwise} \end{cases}$$

Figure 1 depicts the images of (u_i, u_j) and (u''_i, u_j) in utility space. From u_i to u''_i , we adjust the utility for consequences with $u_i(x) \leq u_i(x^*)$ toward $u'_i(x)$ without raising it above $u_i(x^*)$. Setting u_j as we did, we can now apply independence of redundant acts to the corresponding profiles.

Let $\pi_i, \pi_j \in \Pi$ with $\pi_i \neq \pi_j$ and \succsim_i, \succsim_i'' , and \succsim_j be represented by the pairs $(\pi_i, u_i), (\pi_i, u''_i)$, and (π_j, u_j) , respectively. First, since $u_i(x) = u''_i(x)$ for $x \in \{x_0, x_1, x^*\}$, it is clear that $\succsim = (\succsim_i, \succsim_j)$ and $\succsim'' = (\succsim_i'', \succsim_j)$ agree on the preferences over acts in \mathcal{G} . Second, since $\mathbf{u}(x)$

is in the convex hull of $\{\mathbf{u}(x_0), \mathbf{u}(x_1), \mathbf{u}(x^*)\}$ for all $x \in X$, we have that for every act $f \in \mathcal{F}$, there is an act $g \in \mathcal{G}$ such that $f \sim_i g$ and $f \sim_j g$. The analogous assertion holds for \succsim_i'' and \succsim_j . It follows from independence of redundant acts that with $\succsim = \Phi(\succsim)$ and $\succsim'' = \Phi(\succsim'')$, we have for all $g, g' \in \mathcal{G}$, $g \succ g'$ if and only if $g \succ'' g'$. Let (π, u) and (π'', u'') be the beliefs and utility functions associated with \succsim and \succsim'' , respectively. Note that $u(x_0) = u''(x_0) = 0$ and $u(x_1) = u''(x_1) = 1$.

If $\lambda_i(u_i) \neq \lambda_i(u_i'')$, then $\pi \neq \pi''$ since $\pi_i \neq \pi_j$. So we can find an event E such that $\pi(E) = \frac{1}{2} \neq \pi''(E)$. It follows that $x_0Ex_1 \sim x_1Ex_0$ but $x_0Ex_1 \not\sim'' x_1Ex_0$, which is a contradiction since both acts are in \mathcal{G} .

If $\mu_i(u_i) \neq \mu_i(u_i'')$, then $u(x^*) \neq u''(x^*)$, since $u_i(x^*) = u_i'(x^*) \neq u_j(x^*)$. Let E be an event such that $\pi(E) = \pi''(E) = u(x^*)$. Then $x^* \sim x_1Ex_0$ but $x^* \not\sim x_1Ex_0$, which is again a contradiction.

We conclude that $\lambda_i(u_i) = \lambda_i(u_i'')$ and $\mu_i(u_i) = \mu_i(u_i'')$. The function u_i'' is closer to u_i' than is u_i , since we have constructed it by moving utilities toward those in u_i' . Two more modifications of agent 1's utility function along the same lines will result in u_i' . To this end, we apply the same construction first to the profiles with utility functions (u_i'', u_j') and (u_i''', u_j') and then to profiles with utility functions (u_i''', u_j) and (u_i', u_j) (and the same beliefs π_i and π_j).

$$u_i'''(x) = \begin{cases} u_i''(x^*) & \text{if } u_i''(x) \geq u_i''(x^*) \text{ and } u_i'(x) < u_i''(x^*) \\ u_i'(x) & \text{if } u_i''(x) \geq u_i''(x^*) \text{ and } u_i'(x) \geq u_i''(x^*) \\ u_i''(x) & \text{if } u_i''(x) < u_i''(x^*) \end{cases} \quad u_j'(x) = \begin{cases} 1 & \text{if } u_i''(x) \geq u_i''(x^*) \\ u_i''(x) & \text{otherwise} \end{cases}$$

In summary, this gives $\lambda_i(u_i) = \lambda_i(u_i')$ and $\mu_i(u_i) = \mu_i(u_i')$ and proves the lemma. □

Theorem 1 now follows from Proposition 1 and Lemma 11.