Approval Voting under Dichotomous Preferences: A Catalogue of Characterizations

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Approval voting allows every voter to cast a ballot of approved alternatives and chooses the alternatives with the largest number of approvals. Due to its simplicity and superior theoretical properties it is a serious contender for use in real-world elections. We support this claim by giving eight characterizations of approval voting. All our results involve the reinforcement axiom, which requires choices to be consistent across different electorates. In addition, we consider strategyproofness, consistency with majority opinions, consistency under cloning alternatives, and invariance under removing inferior alternatives. We prove our results by reducing them to a single base theorem, for which we give a simple and intuitive proof.

1 Introduction

Around the world, when electing a leader or a representative, plurality is by far the most common voting system: each voter casts a vote for a single candidate, and the candidate with the most votes is elected. In pioneering work, Brams and Fishburn (1983) proposed an alternative system: approval voting. Here, each voter may cast votes for an arbitrary number of candidates, and can thus choose whether to approve or disapprove of each candidate. The election is won by the candidate who is approved by the highest number of voters. Approval voting allows voters to be more expressive of their preferences, and it can avoid problems such as vote splitting, which are endemic to plurality voting. Together with its elegance and simplicity, this has made approval voting a favorite among voting theorists (Laslier, 2011), and has led to extensive research literature (Laslier and Sanver, 2010).

Political scientists have conducted field experiments to evaluate the performance of approval voting in major political elections. Two large-scale experiments are due to

Laslier and Van der Straeten (2004, 2008) during the 2002 French presidential election and Alós-Ferrer and Granić (2012) during the 2008 state election and 2009 federal election in Germany. They report that voters reacted well to approval voting and cast very few invalid ballots. In both instances, the experiments indicate that the election results would have been significantly different under approval voting as compared to the current voting method (plurality with runoff in France and a variant of plurality voting in Germany). In particular, Alós-Ferrer and Granić (2012) found that parties that are perceived as small tend to receive more support under approval voting, presumably because voters cast ballots strategically under plurality voting to avoid "wasting their vote" on a party that has no chance at winning the election.

Approval voting combines two ideas: a simple yet expressive ballot format, and an aggregation method for deciding on a winner given the submitted ballots. Let us consider these two components in turn. First we discuss situations where using approval ballots is appropriate, and then we give reasons why approval voting is the best aggregation method given approval ballots.

Dichotomous preferences We consider the case when the voters have dichotomous preferences. That is, every voter's preferences are given by a partition into approved alternatives and disapproved alternatives, such that the voter is indifferent between all approved alternatives and indifferent between all disapproved alternatives, but strictly prefers each approved alternative to each disapproved alternative. Dichotomous preferences are natural when it is only relevant whether an alternative or candidate meets certain requirements or not.

(i) A group of co-workers aims to schedule a time slot for a meeting. Each of them prefers the slots for which she is available to those where she is unavailable but is otherwise indifferent.

(ii) A hiring committee selects a candidate for performing a clearly defined task. Each member of the committee assesses the candidates and prefers those deemed capable of performing the task to the remaining ones but is otherwise indifferent.

(iii) A company decides on an IT service provider. Each division prefers all providers who offer the services they require to all remaining providers but is otherwise indifferent.

Dichotomous preferences can also arise on behavioral grounds. If it is costly or computationally impractical for a voter to evaluate the alternatives precisely, she may resort to a rough classification into acceptable and unacceptable alternatives. Dichotomous preferences have been considered in the present context of voting (Bogomolnaia et al., 2005), but also for matching (Bogomolnaia and Moulin, 2004) and auction theory (Malik and Mishra, 2021).[1] As in the latter case, dichotomous preferences can be a starting point for a theoretical analysis when the problem is inaccessible on larger preference domains.

[1] In auction theory, some authors have considered single-minded bidders which (assuming free disposal) value a bundle at 1 if it contains a given object and at 0 otherwise (see, e.g., Milgrom and Segal, 2017). Hence, single-mindedness is a special case of dichotomous preferences.

Fishburn (1979)	anon.	neutr.	reinf.[2]	faithfulness[9]	cancellation[12,13]
Alós-Ferrer (2006)			reinf.[2]	faithfulness[9,17]	cancellation[12,13]
Fishburn (1978)	anon.	neutr.[16]	reinf.[2]		disjoint equality[12]
Theorem 1			reinf.[2]	faithfulness[16]	disjoint equality[12]
Theorem 2	anon.[1]	neutr.[3,4,15]	reinf.[2]	non-trivial[10]	strategyproofness[9,5]
Theorem 3			reinf.[6]	continuity[7]	semi-Condorcet consistency[5]
Theorem 4		neutr.[14]	reinf.[6]	continuity[7]	avoid Condorcet losers[5]
Theorem 5		neutr.[16]	reinf.[8]	continuity[7]	majority consistency[5]
Theorem 6	anon.[1]		reinf.[2]	faithfulness[9,10]	clone consistency[5]
Theorem 7	anon.[1]	neutr.[3]	reinf.[2]	faithfulness[9,10]	independence of losers[5]
Theorem 8	anon.[1]	neutr.[3]	reinf.[2]		independence of dominated alt.[5]
Theorem 9	anon.[1]		reinf.[8]	reversal symm.[5]	independence of never-approved alt.[11]

Table 1: List of results characterizing AV. Superscripts indicate the labels of examples in Section 7 showing that the specified axiom cannot be dropped. Axioms without a superscript are redundant. The rows in gray indicate known results.

For dichotomous preferences, the restriction to approval ballots can be justified as follows. By the revelation principle, every social choice function that can be implemented by a dominant strategy incentive-compatible mechanism can be implemented through a direct mechanism that asks voters to report their preferences. Identifying every dichotomous preference relation with the ballot of approved alternatives shows that this ballot format entails no loss of generality for strategyproof mechanisms under dichotomous preferences. A classic result by Brams and Fishburn (1978) shows that approval voting is strategyproof under weak assumptions about how the voters' preferences extend from alternatives to sets of alternatives. Thus, in particular, approval ballots are fully general for approval voting under dichotomous preferences.

Characterizations of approval voting Once we have decided to use approval ballots in an election, the aggregation method is usually taken to be obvious. While the standard method (electing the alternative that was approved on the highest number of ballots) is certainly natural, there are many other conceivable ways of counting approval ballots. For example, we could use a type of cumulative voting, where each voter has a unit weight which is split uniformly among the approved alternatives. Or we might impose a maximum on the number of alternatives that can be approved by a voter, counting ballots that approve too many alternatives as invalid. Or we could declare as winners all alternatives that are Pareto undominated according to the reported approval ballots.

We claim that all alternative aggregation methods fail some of the properties that are often advanced in favor of approval voting, such as its robustness to strategic misrepresentation, its clone-proofness, or its consistent behavior when merging election results of different districts. We provide exhaustive support for this claim by proving a sequence of axiomatic characterizations. Each row of Table 1 corresponds to a result showing that approval voting is the unique aggregation function satisfying the axioms in the row. Taken together, these results provide axiomatic support for the common intuition that approval voting is the uniquely best way to aggregate approval ballots in single-winner elections.

Our results follow a long line of papers that have axiomatically characterized approval voting, starting with the early work of Fishburn (1978, 1979). Those characterizations depend on technical axioms that have limited intuitive appeal. For example, Fishburn (1979) uses "cancellation", which requires that if every candidate is named on the same number of ballots, then the rule should declare a tie between all candidates. Fishburn (1978) uses "disjoint equality", which prescribes that for two voters with disjoint approval ballots, the ballot aggregation function should declare a tie between all candidates in the union of their ballots. In place of these axioms, our characterizations use properties like strategyproofness, clone-proofness, or avoidance of Condorcet losers, which we think are easier to defend.

Technically, the common basis for all our characterizations is the reinforcement axiom, which requires that the ballot aggregation function makes consistent choices across different sub-electorates.[2] Imagine, for example, that a nation is split into several states, and suppose that there exists a candidate who wins in every state (when counting only the ballots cast in that state). Reinforcement requires that, when counting all ballots nationwide, the ballot aggregation function elects exactly those candidates who win in every state individually. This axiom applies to ballot aggregation functions defined for varying numbers of voters, and thus we operate in a framework with variable electorates. Reinforcement is known to be the driving force in many characterizations of scoring-based rules in social choice theory (see, e.g., Young, 1975; Young and Levenglick, 1978; Myerson, 1995). In contrast to some other characterization results based on reinforcement, our proofs use only elementary mathematics.[3] The appeal of direct and elementary proofs is not merely aesthetic; this property allows our characterizations to be used to *explain* the election outcome to voters: given a specific profile of approval ballots, one can automatically produce a short (polynomial-length) proof showing that the axioms imply that exactly the winners of approval voting need to be elected in the given profile.[4]

We will not prove our characterization results from scratch each time. Instead, we will prove them by reduction to a single base theorem (with the exception of Theorem 3), which characterizes approval voting as the unique rule satisfying reinforcement, faithfulness, and disjoint equality. This base theorem strengthens the result of Fishburn

[2]This axiom was introduced by Young (1975), who called it "consistency". We use "reinforcement" (Moulin, 1988a,b; Young, 1988; Myerson, 1995) to distinguish it from other consistency axioms such as those for variable agendas which we consider in Section 6.

[3]For example, Young (1975); Fishburn (1979); Myerson (1995) use separating hyperplane theorems for their characterizations of scoring-based functions and Pivato (2013) uses results from group theory for this generalization of Myerson's result.

[4]The result of Theorem 2 is explainable in this way when strengthening non-triviality to faithfulness. The results of Theorems 4 and 5 use a simple limit argument, and so would require a stronger logic than in other cases. Previously, Cailloux and Endriss (2016) showed that the Borda rule can be similarly explained in terms of the axioms of Young's (1974) characterization.

(1978, 1979) by avoiding the use of any symmetry arguments based on neutrality. (The simple proof is very short and may be of interest for teaching purposes.) We then prove the remaining characterizations by showing that a ballot aggregation function satisfying the axioms will also satisfy faithfulness and disjoint equality, and hence we are done by invoking the base theorem. We take care to ensure that all of our results are axiomatically tight, and in Section 7 we construct 17 example rules which show that none of the axioms can be dropped or significantly weakened.

After establishing the base theorem, we consider strategic incentives. We characterize approval voting using its well-known property of not being susceptible to strategic misrepresentation of preferences. Our characterization makes weak assumptions about the voters' preferences over sets of alternatives (following Kelly, 1977), though we show that approval voting in fact satisfies significantly stronger strategyproofness notions. We then turn to axioms that require a rule to be consistent with the will of a majority of the voters, and show that approval voting can be characterized either using the fact that it never elects a Condorcet loser, or that it only elects candidates with a majority support in cases where more than half the voters submit the same approval ballot. Finally, we characterize approval voting by its resistance to the spoiler effect, which is familiar from plurality voting where the presence of a weak candidate can change the winner by 'splitting the vote'. We formalize resistance to the spoiler effect in four different ways – independence axioms and Tideman's (1987) cloning consistency – and show that each characterizes approval voting.

In Section 8, we discuss other works on characterizing approval voting. Of particular note is Fishburn's (1979) paper, which shows that neutrality and reinforcement characterizes a class of scoring rules. Fishburn then shows that the only scoring rule satisfying disjoint equality or strategyproofness is approval voting. We obtain these results more directly, without reasoning about scoring rules. In the appendix, we state omitted proofs.

2 The Model

Let X be a finite set of alternatives and $\mathcal{A}$ be the set of non-empty subsets of X. A preference relation on X is a complete, reflexive, and transitive relation on X. It is dichotomous if it has at most two indifference classes. Thus, every dichotomous preference relation can be identified with the set $R \in \mathcal{A}$ of alternatives in the indifference class of most-preferred alternatives. We say a voter with dichotomous preferences R approves the alternatives in R and disapproves the alternatives in $X \setminus R$.

The set of admissible ballots is also $\mathcal{A}$. A ballot profile P is a function from the ballot set $\mathcal{A}$ to the non-negative integers such that $\sum_{A\in\mathcal{A}} P(A) > 0$. We interpret $P(A)$ as the number of voters whose ballot is A. The approval score $P[a]$ of an alternative a is the number of voters whose ballot includes a, so $P[a] = \sum_{A\in\mathcal{A}:a\in A} P(A)$. Often, it will be useful to identify elements of $\mathcal{A}$ with single-voter ballot profiles. For example, $P + A$ is the profile resulting from P by adding one voter with ballot A; similarly, the profile $P + kA$ is obtained by adding k voters with ballot A to P. For a permutation π on X, the profile $\pi(P)$ has $P(A)$ voters with ballot $\pi(A)$ for every ballot A.

A ballot aggregation function f maps each profile P to a set of winning alternatives $f(P) \in \mathcal{A}$. Typically, $f(P)$ will be a singleton, but f may sometimes declare several alternatives to be tied. Our definition of ballot profiles entails that ballot aggregation functions are anonymous, since they cannot distinguish between voters submitting the same ballot.[5]

We recall a number of axioms for ballot aggregation functions from the literature. A ballot aggregation function satisfies each of the following axioms if the corresponding property holds for all profiles P, P', ballots A, B, and permutations π.

$$f(P) \cap f(P') = f(P + P') \quad \text{whenever } f(P) \cap f(P') \neq \emptyset \qquad \text{(reinforcement)}$$
$$f(A) = A \qquad \text{(faithfulness)}$$
$$f(\pi(P)) = \pi(f(P)) \qquad \text{(neutrality)}$$
$$f(A + B) = A \cup B \quad \text{whenever } A \cap B = \emptyset \qquad \text{(disjoint equality)}$$
$$f(P) = X \quad \text{whenever } P[a] = P[b] \text{ for all } a, b \in X \qquad \text{(cancellation)}$$

We are interested in the ballot aggregation function called *approval voting* (AV), which chooses all alternatives with maximal approval score. It is elementary to check that AV satisfies all of the axioms above. We will also refer to the trivial function $TRIV$ selecting all alternatives in all profiles, and the function $-AV$ selecting all alternatives with minimal approval score. A ballot aggregation function is non-trivial if it is not $TRIV$.

3 Base Theorems

We begin by proving our base theorem: approval voting is the only ballot aggregation function satisfying reinforcement, disjoint equality, and faithfulness. We will use this base theorem to obtain the results in Sections 4–6. Fishburn (1978) proves the same result with neutrality in place of faithfulness (see also Fishburn, 1979, Theorem 5). Another characterization of Fishburn (1979) uses neutrality, reinforcement, and *cancellation*. Here, too, it is possible to prove the analogous result with faithfulness instead of neutrality, and Alós-Ferrer (2006) gives a simple proof.[6] For our purposes, a base theorem with disjoint equality is more useful than one with cancellation. The first two sections in Table 1 give an overview of these results. Lemma 1 in the appendix shows how the results of Fishburn (1978) and Alós-Ferrer (2006) can be obtained from Theorem 1.

Our proof proceeds as follows. Given an arbitrary profile, consider an approval winner, say a, and an alternative chosen by a rule that satisfies the axioms, say b. Based on these two alternatives, we construct an auxiliary profile in which the rule also chooses b.

[5] One can verify that the characterizations in Sections 3 and 5 continue to hold when allowing non-anonymous ballot aggregation functions. In other results, anonymity is a necessary assumption (see Section 7).

[6] In a survey article, Xu (2010, Theorem 5.3.2) points out that the proof of Alós-Ferrer (2006) can be adapted to give a characterization with disjoint equality, though this adaptation implicitly requires using a stronger version of disjoint equality that applies to both two-voter and three-voter profiles.

Adding this profile to the starting profile yields a larger profile in which (by reinforcement) our rule still chooses b. Moreover, it can only choose a if it also did so in the starting profile. Decomposing the large profile in a different way shows that the rule has to choose both a and b. Hence, it indeed had to choose a in the starting profile, and so it has to choose all approval winners. Showing that it cannot choose any additional alternatives is similar.

Theorem 1. *AV is the only ballot aggregation function satisfying reinforcement, disjoint equality, and faithfulness.*

Proof. Let P be a profile. If some alternative is approved by all voters, i.e., $\bigcap_{A\in\mathcal{A}:P(A)\geqslant 1} A \neq \emptyset$, then faithfulness and reinforcement imply that $f(P) = \bigcap_{A\in\mathcal{A}:P(A)\geqslant 1} A = AV(P)$ and we are done. We call such a profile P a consensus profile.

Now consider the case that P is not a consensus profile. Let $a \in AV(P)$ and $b \in f(P)$. We will show that $a \in f(P)$ and $b \in AV(P)$. If $a = b$ this is obvious, so assume that they are distinct. Let $P[a/b]$ be the number of voters in P who approve a (and possibly other alternatives) but not b, and let $P[b/a]$ be the number of voters in P who approve b (and possibly other alternatives) but not a. Moreover, let $P[\cdot/ab]$ be the number of voters in P who approve neither a nor b. Since P is not a consensus profile, at least one voter has to disapprove a, and so $P[a/b] + P[b/a] + P[\cdot/ab] \geqslant P[b/a] + P[\cdot/ab] > 0$. Let P' be the profile on $P[a/b] + P[b/a] + P[\cdot/ab]$ voters such that

$$P'(\{a\}) = P[b/a], \qquad P'(\{b\}) = P[a/b], \quad \text{and} \quad P'(\{a,b\}) = P[\cdot/ab].$$

In the following, we will consider the profile $P + P'$, and decompose it in two ways.

In the first decomposition, we pair each voter in P (except those approving both a and b) with a voter in P' who approves a disjoint set of candidates:

$$P+P' = \sum_{\substack{A\in\mathcal{A}\\ a\in A, b\notin A}} P(A)\cdot(A+\{b\}) + \sum_{\substack{A\in\mathcal{A}\\ b\in A, a\notin A}} P(A)\cdot(A+\{a\}) + \sum_{\substack{A\in\mathcal{A}\\ a,b\notin A}} P(A)\cdot(A+\{a,b\}) + \sum_{\substack{A\in\mathcal{A}\\ a,b\in A}} P(A)\cdot A.$$

This pairing allows us to apply disjoint equality to each term of the first three sums, and we see that f elects both a and b in each of them. By faithfulness, we obtain the same conclusion for the terms of the fourth sum. reinforcement implies that $a, b \in f(P + P')$.

In the second decomposition, we pair each $\{a\}$-voter in P' with a $\{b\}$-voter in P'. Since $a \in AV(P)$, we have $P[a/b] \geqslant P[b/a]$, so each $\{a\}$-voter can be matched:

$$P + P' = P + P[b/a] \cdot (\{a\} + \{b\}) + (P[a/b] - P[b/a]) \cdot \{b\} + P[\cdot/ab] \cdot \{a, b\}.$$

Considering each term of the sum on the right-hand side separately, we see that f elects b in each of them: $b \in f(P)$ by assumption, $f(\{a\} + \{b\}) = \{a, b\}$ by disjoint equality, and $f(\{b\}) = \{b\}$ and $f(\{a, b\}) = \{a, b\}$ by faithfulness.

If $a \notin f(P)$ then reinforcement applied to the second decomposition implies that $a \notin f(P + P')$, a contradiction to $a, b \in f(P + P')$. If $b \notin AV(P)$, then $P[a] > P[b]$ and thus $P[a/b] - P[b/a] > 0$, so that the third term in the sum does not vanish. Hence by reinforcement $f(P + P') = \{b\}$, again contradicting $a, b \in f(P + P')$. So $a \in AV(P)$ implies $a \in f(P)$, and $b \in f(P)$ implies $b \in AV(P)$. Hence $f(P) = AV(P)$. □

Remark 1. Theorem 1 also holds for the weakening of disjoint equality that only requires $f(A + B) \supseteq A \cup B$ for all disjoint ballots A and B. The proof can be copied almost verbatim. □

The goal of the results in the following sections is to replace disjoint equality and cancellation by axioms of one of three types: resistance to strategic misrepresentation of preferences, majoritarian properties (such as never electing Condorcet losers), and properties that ensure consistency across different agendas (such as independence of unchosen alternatives). Theorem 1 will be useful throughout, since we will prove the subsequent characterizations by reducing them to this basic result.

4 Strategyproofness

In this section, we study when ballot aggregation functions incentivize voters to cast their ballots sincerely. The sincere ballot for a voter with dichotomous preferences is to cast the ballot of alternatives she approves.[7] Since ballot aggregation functions return sets of alternatives, the incentives of voters depend on their preferences over sets. We thus use preference extensions from dichotomous preferences over alternatives to preferences over sets of alternatives. First, we consider Kelly's extension, which leads to coarse preferences over sets. We show that approval voting is the only ballot aggregation function satisfying reinforcement and neutrality for which voters can never gain by casting an insincere ballot. Second, we show that for a finer extension, called Fishburn's extension, it is a weakly dominant strategy under approval voting for voters to cast their ballots sincerely.

Consider a voter who approves the set R of alternatives. According to Kelly's (1977) extension, she weakly prefers the set $Y \in \mathcal{A}$ over $Z \in \mathcal{A}$ if she weakly prefers every alternative in Y to every alternative in Z (see also Brandt, 2015; Brandt et al., 2020). For her dichotomous preferences, this happens if either Y consists only of approved alternatives of if Z does not contain any approved alternatives. Denoting by $\succsim_R^K$ the resulting preferences over sets, we have

$$Y \succsim_R^K Z \text{ if and only if } Y \subseteq A \text{ or } Z \cap A = \emptyset.$$

A ballot aggregation function f is called *Kelly-manipulable* if for some voter with approval set R, there is a ballot profile P with $f(P+A) \succ_R^K f(P+R)$ for some non-sincere ballot $A \in \mathcal{A}$. In particular, reporting the sincere ballot is not a best response at P. In this case, reporting the insincere ballot A ensures that either all winning alternatives are approved instead of having some disapproved alternatives in the winning set, or at least one approved alternative is in the winning set rather than only disapproved alternatives.

[7] Brams and Fishburn (1978) consider linear preferences over alternatives and call a ballot sincere if it is the upper contour set of some alternative. Alós-Ferrer and Buckenmaier (2019) extend sincerity to weak preferences in two ways. They call a ballot sincere if it is "upward closed" (any alternative strictly preferred to an alternative on the ballot is also on the ballot) and strongly sincere if it is the upper contour set of some alternative. For dichotomous preferences, our notion of sincerity equals their strong sincerity.

Kelly's called these "clear manipulations" since no matter which tie-breaking mechanism is invoked to select a final outcome from choice sets, manipulating is always at least as good as truth-telling and strictly better for some tie-breaking mechanism. Another interpretation is based on expected utilities. Suppose approved alternatives have utility 1 and disapproved alternatives have utility 0. Suppose further that ties among winning alternatives are broken by some lottery with positive probability for each winning alternative, but that this lottery is unknown to the voters. Then a voter can Kelly-manipulate if and only if reporting some insincere ballot has higher expected utility than truth-telling for all possible tie-breaking lotteries. Brandt et al. (2020) elaborate on this interpretation in detail.

It is not hard to see that AV is never Kelly-manipulable (see Proposition 1 below). By contrast, Brandt (2015) showed that every Condorcet-consistent social choice correspondence is Kelly-manipulable if preferences are not dichotomous. In Theorem 2, we characterize AV using Kelly-manipulability. Based on his discussion of scoring rules, Fishburn (1979, Theorem 10) obtains a similar characterization using a more restrictive notion of manipulability, though his proof does not require the extra strength.

Theorem 2. *AV is the only non-trivial ballot aggregation function satisfying reinforcement and neutrality that is not Kelly-manipulable.*

Proof. We prove that any such f satisfies faithfulness and disjoint equality. Theorem 1 then implies $f = AV$.[8]

First we show faithfulness. Neutrality implies that $f(A) \in \{X, A, X \setminus A\}$ for all ballots A. If $f(A) = A$ for all A, then f satisfies faithfulness and there is nothing left to show. If $f(A) = X$ for all A, reinforcement implies that $f(P) = X$ for all profiles P, i.e., $f = TRIV$, which is contrary to the assumption that f is non-trivial. If there is a ballot A such that $f(A) = X \setminus A$, then a voter who approves A can Kelly-manipulate by reporting the ballot X at the empty profile since $f(X) = X$ and $X \succ_A^K (X \setminus A)$. In the remaining case, there are ballots A, B such that $f(A) = A$ and $f(B) = X$. If $|B| > |A|$, let $B' \subseteq B$ be a ballot with $|B'| = |A|$. By neutrality, $f(A) = A$ implies $f(B') = B'$. So a voter who approves B can Kelly-manipulate by reporting B' at the empty profile since $B' \succ_B^K X$, which is a contradiction. Thus, by neutrality, there is $k \in \{2, \dots, m-1\}$ such that $f(A) = A$ for all A with $|A| \geqslant k$ and $f(A) = X$ for all A with $|A| \leqslant k-1$. Let A be a ballot such that $|A| = k-1 \leqslant m-2$ and $a, b \in X \setminus A$ be two distinct alternatives. Then, $f(A) = X$, $f(A \cup \{a\}) = A \cup \{a\}$, and $f(A \cup \{b\}) = A \cup \{b\}$. Hence, by reinforcement, $f(A \cup \{a\} + A) = A \cup \{a\}$ and $f(A \cup \{a\} + A \cup \{b\}) = A$. Thus, a voter who approves A can Kelly-manipulate by reporting $A \cup \{b\}$ at the profile $A \cup \{a\}$, since $A \succ_A^K A \cup \{a\}$.

Second we show disjoint equality. To this end, let A, B be two disjoint ballots. We first show that $f(A + B) \subseteq A \cup B$. Assume for contradiction that this is not the case, i.e., $f(A + B) \setminus (A \cup B) \neq \emptyset$. Let $b \in B$, $c \in f(A + B) \setminus (A \cup B)$, and $C = B \setminus \{b\} \cup \{c\}$.

[8] By invoking Fishburn's (1978) characterization instead of Theorem 1, it would suffice to show that f satisfies disjoint equality. However, our proof for disjoint equality uses faithfulness, so we would still need to establish faithfulness.

By faithfulness, we have $f(C) = C$. Then, reinforcement implies that $f(A+B+C) = f(A+B) \cap C$. Hence, $b \notin f(A+B+C)$ and $c \in f(A+B+C)$. Since $|B| = |C|$ and $A \cap B = A \cap C = \emptyset$, this contradicts neutrality.

So neutrality implies $f(A+B) \in \{A \cup B, A, B\}$. Assume for contradiction that $f(A+B) = A$. (The case $f(A+B) = B$ is analogous.) Let $a \in A$ and $b \in B$. Neutrality and the fact that $f(A+\{b\}) \subseteq A \cup \{b\}$ imply that $f(A+\{b\}) \in \{A \cup \{b\}, A, \{b\}\}$. Since $f(A+B) \cap B = \emptyset$, it follows that $f(A+\{b\}) \cap B = \emptyset$, as otherwise the voter who approves B can Kelly-manipulate at the profile A by reporting $\{b\}$. Hence, $f(A+\{b\}) = A$. Faithfulness implies that $f(\{a,b\}) = \{a,b\}$. Thus, by reinforcement, $f(A+\{b\}+\{a,b\}) = A \cap \{a,b\} = \{a\}$. Then $f(\{a\}+\{b\}+\{a,b\}) = \{a\}$, as otherwise a voter who approves $\{a\}$ can Kelly-manipulate at the profile $\{b\}+\{a,b\}$ by reporting A. This contradicts neutrality.

In summary, we have that f satisfies reinforcement, disjoint equality, and faithfulness, and so $f = AV$ by Theorem 1. □

Remark 2. Our definition of manipulability only considers unilateral deviations. If we strengthen it to allow for deviations by groups of voters, AV turns out to be Kelly-manipulable. For example, in the profile $P = \{a\}+\{b\}+2\{c\}$, AV chooses $\{c\}$. If the voters with ballots $\{a\}$ and $\{b\}$ report $\{a,b\}$ instead, we obtain the profile $P' = 2\{a,b\}+2\{c\}$ for which AV returns $\{a,b,c\}$. Hence, there exists a manipulation for two voters who approve $\{a\}$ and $\{b\}$, respectively, where they obtain some approved alternative instead of only disapproved alternatives. Examples where voters obtain only approved alternatives instead of some disapproved alternative can be constructed likewise. Thus, from Theorem 2, we see that every non-trivial ballot aggregation function satisfying reinforcement and neutrality is Kelly-manipulable by a group of voters.[9]

We used a weak definition of incentive-compatibility in Theorem 2, requiring that an insincere ballot can never make the output unambiguously better. This notion does not rule out that, for example, an insincere ballot adds approved alternatives to the output while removing others. We now consider Fishburn's extension, which is a refinement of Kelly's extension. That is, it assumes that voters can compare more sets of alternatives. We prove that truth-telling is a weakly dominant strategy for these preferences under AV.

Consider a voter who approves the set R of alternatives. Assume that the voter weakly prefers the set $Y \in \mathcal{A}$ to $Z \in \mathcal{A}$ if she weakly prefers every alternative in $Y \setminus Z$ to every alternative in $Y \cap Z$ to every alternative in $Z \setminus Y$. Denoting by $\succsim_R^F$ the resulting preferences over sets, we have

$$Y \succsim_R^F Z \text{ if and only if } \begin{cases} Y \subset R, \text{ or} \\ Z \cap R = \emptyset, \text{ or} \\ Y \setminus Z \subseteq R \text{ and } (Z \setminus Y) \cap R = \emptyset. \end{cases}$$

[9]Brandt et al. (2020, Remark 2) show that the ballot aggregation function returning all Pareto undominated alternatives is never Kelly-manipulable by a group of voters. However, it violates reinforcement.

Fishburn (1972) motivated this preference extension as arising as the preferences of an expected utility maximizing voter when ties between alternatives are broken as follows. Suppose there is a fixed prior probability distribution with full support over the alternatives, which is unknown to the voters. Ties are broken by conditioning this distribution on the set of winning alternatives. Then an agent prefers one set to another according to Fishburn's extension if and only if the former has higher expected utility than the latter for any prior distribution.

We say that a ballot aggregation function f is *Fishburn-strategyproof* if truth-telling is a weakly dominant strategy for Fishburn's extension. That is, for all profiles P and all ballots $A \in \mathcal{A}$, $f(P+R) \succsim_R^F f(P+A)$. Thus, either the sincere ballot yields only approved alternatives, the insincere ballot yields only disapproved alternatives, or, if neither of those holds, the insincere ballot can only remove approved alternatives and add disapproved alternatives.

For non-dichotomous preferences, this notion of strategyproofness is unduly restrictive.[10] However, under dichotomous preferences, AV satisfies it. This can be deduced from the theory developed by Brams and Fishburn (1978, Theorem 4); here, we give a direct proof.

Proposition 1. *AV is strategyproof for Fishburn's extension.*

Proof. Consider a profile P, a dichotomous preference R, and a ballot A. Assume that neither $AV(P+R) \not\subseteq R$ nor $AV(P+A) \cap R \neq \emptyset$. For $a \in AV(P+R) \cap (X \setminus R)$ and $b \in AV(P+A) \cap R$, we have

$$(P+R)[a] \leqslant (P+A)[a] \leqslant (P+A)[b] \leqslant (P+R)[b] \leqslant (P+R)[a].$$

Hence, $(P+R)[a] = (P+A)[b]$ for all $a \in AV(P+R)$ and $b \in AV(P+A)$, meaning that the approval score of approval winners is the same in both profiles. So $AV(P+R) \cap (X \setminus R) \subseteq AV(P+A)$ and $AV(P+A) \cap R \subseteq AV(P+R)$, which is the third case in the definition of Fishburn's extension. □

Alós-Ferrer and Buckenmaier (2019) consider sincere ballots under approval voting when voters have arbitrary transitive preferences over the alternatives. They show that voters always have a (strongly) sincere ballot that is a best-response against the other voters' ballots assuming that the preference extension satisfies three conditions. The first, known as condition (R) from Brams and Fishburn (1978), requires that adding alternatives to a set that are preferred to all alternatives, and removing alternatives from a set that are less preferred than all remaining alternatives both constitute weak improvements. The other two conditions consider deletion and replacement of alternatives. For dichotomous preferences, they are implied by condition (R). Hence, for dichotomous preferences, casting the unique (strongly) sincere ballot is a weakly dominant strategy for every voter. Since Fishburn's extension satisfies condition (R), their result also implies Proposition 1.

[10]For example, it is incompatible with anonymity and Pareto optimality (Brandt et al., 2018).

In the rest of the paper, we assume that voters' ballots coincide with their preferences. For approval voting, we justify this by the fact that the sincere ballot is a weakly dominant strategy for Fishburn's extension (Proposition 1). In fact, for every insincere ballot, there is a ballot profile for the remaining voters so that the winning set under the sincere ballot is preferred to the winning set under the insincere ballot for Fishburn's extension.

5 Majoritarian Properties

In many democratic contexts, an important goal of voting is to uncover the will of a majority. If the preferences of a majority of voters share a certain feature, then this should be reflected in the collective decision. A classic example of this desideratum is the Condorcet criterion of choosing a Condorcet winner whenever it exists. Other properties in a majoritarian spirit are never choosing a Condorcet loser and choosing a majority winner whenever it exists. We will give a characterization of AV for each of these three properties.

When preferences are dichotomous, the majority relation is transitive (Inada, 1969) and coincides with the relation obtained by ordering alternatives according to their approval score. The maximal elements of the majority relation are the approval winners under sincere voting and preferred to every other alternative by a weak majority of voters (so they are weak Condorcet winners). If there is a unique approval winner, it is a Condorcet winner. Thus, AV is Condorcet consistent. So AV is characterized by Condorcet consistency except for the case of ties at the top of the majority relation. A less demanding notion is Fishburn's (1979) *semi-Condorcet consistency*, which requires that a Condorcet winner always has to be *among* the winners. In his Theorem 6 he shows that AV is the only non-trivial, neutral, and continuous ballot aggregation function satisfying reinforcement and semi-Condorcet consistency. In the context of voting with linear orders rather than approval ballots, these axioms are incompatible (Young and Levenglick, 1978, Theorem 2).

The *continuity axiom* requires that if a is chosen uniquely in a profile P but not in P', then a is again chosen uniquely once we add enough copies of P to P'. Formally, f is *continuous* if for any two profiles P, P' with $f(P) = \{a\}$, there is an integer k such that $f(P' + kP) = \{a\}$.[11] Myerson (1995) calls this property the *overwhelming majority* axiom.

The following theorem shows that neutrality is not required in Fishburn's theorem, which he leaves as an open question.[12] Unlike our other theorems, we do not prove this by reducing it to our base theorem, but prove it from scratch. Interestingly, reinforcement and semi-Condorcet consistency suffice to show that the winners have to be a subset

[11] In the presence of reinforcement, this definition is equivalent to requiring that $a \in f(P' + kP)$ for some k. This is because if $a \in f(P' + kP)$, then $f(P' + (k+1)P) = f(P' + kP) \cap f(P) = \{a\}$ by reinforcement.

[12] Fishburn's continuity and semi-Condorcet consistency are stronger than our notions. His continuity also applies to cases where more than a single alternative is chosen in the duplicated profile; moreover, his semi-Condorcet consistency also applies to weak Condorcet winners.

of the approval winners. So our Example 7 showing that continuity is required for Theorem 3 cannot be improved to a rule that is not a subset of AV.

Theorem 3. *AV is the only non-trivial ballot aggregation function satisfying reinforcement, continuity, and semi-Condorcet consistency.*

In a profile P, an alternative a is a *Condorcet loser* if for every other alternative b, more voters prefer b to a than prefer a to b. A ballot aggregation function *avoids Condorcet losers* if it never chooses a Condorcet loser: $a \notin f(P)$. This is weaker than Condorcet consistency, and hence AV satisfies it. Some refinements of AV satisfy it as well (see Example 3 in Section 7), and thus AV is not characterized by reinforcement, neutrality, and avoidance of Condorcet losers. We can pin down AV uniquely by adding continuity. Together, the four axioms characterize AV. In the context of voting with linear orders, the same axioms characterize Borda's rule (combining results of Smith, 1973 and Young, 1975).

Theorem 4. *AV is the only ballot aggregation function that satisfies reinforcement, neutrality, continuity, and avoids Condorcet losers.*

Proof. We show that any such ballot aggregation function satisfies faithfulness and cancellation and then apply the result of Alós-Ferrer (2006) (also proven in Lemma 1(ii) in the appendix). Faithfulness is easy to see from neutrality and avoiding Condorcet losers. Assume for contradiction that f does not satisfy cancellation. So there is a profile P such that $P[a] = P[b]$ for all alternatives a, b and $f(P) \neq X$. Let $a \in f(P)$. We now construct a profile P_a such that all alternatives have the same approval score in P_a and $f(P_a) = \{a\}$. To this end, let $\Pi_a = \{\pi \in \Pi(X) \colon \pi(a) = a\}$ be the set of all permutations on X that hold a fixed and $P_a = \sum_{\pi \in \Pi_a} \pi(P)$ the profile obtained by summing up all permutations of P for permutations in Π_a. By neutrality, $a \in f(\pi(P))$ for all π and so

$$f(P_a) = \bigcap_{\pi \in \Pi_a} f(\pi(P)) = \bigcap_{\pi \in \Pi_a} \pi(f(P)) = \{a\},$$

where the first equality follows from repeated application of reinforcement, the second equality from neutrality of f, and the third equality from the assumption that $f(P) \neq X$. Then, continuity implies that there is $k \in \mathbb{N}$ such that $f(X \setminus \{a\} + kP_a) = f(P_a) = \{a\}$. However, a is a Condorcet loser in the profile $X \setminus \{a\} + kP_a$, which contradicts the assumption that f avoids Condorcet losers. □

Majority consistency requires that if more than half of the voters in a profile P submit the same ballot A, then at least one alternative from A is a winner (possibly among other alternatives not from A). Notice that every alternative not in A is disapproved by a majority of voters. In such situations, AV will return a subset of A. For voting with linear orders, similar axioms characterize the plurality scoring rule (Lepelley, 1992, see also Sanver, 2002).

Theorem 5. *AV is the only non-trivial ballot aggregation function satisfying reinforcement, neutrality, continuity, and majority consistency.*

6 Consistency Across Variable Agendas

The plurality rule is commonly criticized as suffering from the "spoiler effect", whereby the presence of weak candidates causes an otherwise strong candidate to lose, by "splitting the vote". This kind of problem affects many voting rules besides plurality rule, but is arguably largely avoided by approval voting. To make this claim formal, it is useful to adopt a setting where the set of alternatives (the *agenda*) may vary, so that we can reason about the effect of adding or removing alternatives. In this section, we consider a variety of consistency conditions for varying agendas that are satisfied by approval voting. These conditions capture various senses in which approval voting is robust against the introduction of weak or redundant ("clone") candidates, and hence is immune to the spoiler effect. In addition, these axioms can be seen as ensuring that collective choices are rationalizable, and they can be interpreted as preventing certain manipulations by someone with agenda-setting power.

Let us formally set up the variable-agenda model. We re-interpret X as the set of all potential alternatives. The set of *agendas* $\mathcal{P}(X)$ consists of all non-empty subsets of X. Given an agenda $Y \in \mathcal{P}(X)$, the corresponding ballot set $\mathcal{A}_Y$ contains all non-empty subsets of Y. As before, a profile P on an agenda Y is a function from $\mathcal{A}_Y$ to the non-negative integers such that $\sum_{A \in \mathcal{A}_Y} P(A) > 0$. The restriction of a profile P on agenda Y to agenda $Z \subseteq Y$, written P_Z, is defined by $P_Z(B) = \sum_{A \in \mathcal{A}_Y :\, A \cap Z = B} P(A)$ for all $Z \in \mathcal{A}_Z$. A ballot aggregation function f_Y *on an agenda* Y maps a profile on Y to a subset of Y. A ballot aggregation function $f = (f_Y)_{Y \in \mathcal{P}(X)}$ specifies a ballot aggregation function for each agenda. All axioms defined earlier hold if they hold within each agenda. We will sometimes abuse notation by applying f_Z to a profile P on a larger agenda $Y \supseteq Z$. In such cases, P is to be understood as its restriction P_Z to Z.

In the context of linear orders, Tideman (1987) noticed that under many common voting rules, the winner can change when some candidate is 'cloned' by introducing new candidates that each voter ranks in an adjacent position to the original candidate. Few voting rules can avoid this behavior. However, in the context of approval ballots instead of linear orders, AV is a ballot aggregation function that does avoid it. To define an appropriate version of Tideman's axiom for dichotomous preferences, we say that two alternatives a, b are *clones* of each other in a profile P if every voter is indifferent between a and b, or in other words either approves both a and b or disapproves both. A ballot aggregation function f satisfies *clone consistency* if, whenever a and b are clones in P, then $f_{Y \setminus \{b\}}(P) = f_Y(P) \cap (Y \setminus \{b\})$ and $b \in f_Y(P)$ if and only if $a \in f_{Y \setminus \{b\}}(P)$. Thus, adding a clone b of an alternative a to a profile has no effect on whether other alternatives are chosen or not, and b is chosen if and only if a was chosen in the original profile.

It turns out that reinforcement, clone consistency, and faithfulness characterize AV. In fact, using a more technical argument (omitted here), one can show that the only rules that satisfy reinforcement and clone consistency are AV, $-AV$, and $TRIV$. Thus, faithfulness is only required to rule out $-AV$ and $TRIV$.

Theorem 6. *AV is the only ballot aggregation function satisfying reinforcement, clone consistency, and faithfulness if $|X| \geqslant 4$.*

The next axiom prevents a losing alternative from spoiling the election. Taken from choice theory, it prescribes that removing unchosen alternatives from the agenda should not change the choice set (see, e.g., Chernoff, 1954; Aizerman and Aleskerov, 1995; Brandt and Harrenstein, 2011). Formally, a ballot aggregation function f satisfies *independence of losers* if $f_Y(P) = f_Z(P)$ for all profiles P and agendas $Z \subseteq Y$ with $f_Y(P) \subseteq Z$. Independence of losers prevents, for example, a candidate who has no chance at winning from influencing the election by withdrawing from the election. Because removing losers does not change the approval scores of other candidates, AV satisfies this property, and AV can be characterized by reinforcement, neutrality, faithfulness, and independence of losers. When dropping faithfulness, at the expense of a more technical proof, one can show that AV, $-AV$, and $TRIV$ are the only rules that satisfy these axioms.

Theorem 7. *AV is the only ballot aggregation function satisfying reinforcement, neutrality, faithfulness, and independence of losers.*

Proof. Take any such ballot aggregation function f and some agenda $Y \in \mathcal{P}(X)$. We will omit the subscript Y for f within this proof. We show that f satisfies disjoint equality and apply Theorem 1.

Let A, B be two disjoint ballots and $a \in A$ and $b \in B$. Neutrality implies that $f(A + B) \in \{X, A \cup B, A, B, X \setminus B, X \setminus A\}$. First assume for contradiction that $f(A + B) \not\subseteq A \cup B$ and let $c \in f(A + B) \setminus (A \cup B)$. By faithfulness, we have $f(\{c\}) = \{c\}$. Hence, by reinforcement, $f(A + B + \{c\}) = \{c\}$. Independence of losers implies that $f(A + B + \{c\}) = f_{\{a,b,c\}}(\{a\} + \{b\} + \{c\}) = \{c\}$, which contradicts neutrality. So we have $f(A + B) \in \{A \cup B, A, B\}$.

Second, consider the case $f(A + B) = A$. From the previous case and neutrality, we know that $f(\{a\} + \{b\}) = \{a, b\}$. Hence, by reinforcement, $f(A + B + \{a\} + \{b\}) = \{a\}$. Independence of losers implies that $f(A+B+\{a\}+\{b\}) = f_{\{a,b\}}(\{a\}+\{b\}+\{a\}+\{b\}) = \{a\}$, which again contradicts neutrality. Similarly, we get a contradiction if $f(A+B) = B$. Hence, $f(A+B) = A\cup B$ is the only possibility, and thus f satisfies disjoint equality. □

To decide whether removal of an alternative is allowed to change the outcome, independence of losers references the choice set of the ballot aggregation function under consideration. Alternatively, we can look for a more objective approach to identify inferior alternatives whose removal should not change the election outcome. We will consider two such notions. The first requires that the removal of a Pareto dominated alternative does not change the set of winners. The second is weaker and only requires that removing an alternative which is not approved by any voter does not change the outcome.

An alternative y is Pareto dominated in a profile P if there exists an alternative x such that every voter in P who approves y also approves x, and there is a voter in P who approves x but not y. This is the natural definition under dichotomous preferences. A ballot aggregation function satisfies *independence of Pareto dominated alternatives* if $f_Z(P) = f_Y(P)$ for all agendas Y and Z and profiles P on Y such that $Z \subseteq Y$ and all alternatives in $Y \setminus Z$ are Pareto dominated in P. In conjunction with reinforcement and neutrality, this characterizes AV. In the context of linear orders, these axioms characterize the plurality scoring rule (Richelson, 1978; Ching, 1996; Öztürk, 2020).

Theorem 8. *AV is the only ballot aggregation function satisfying reinforcement, neutrality, and independence of Pareto dominated alternatives.*

A weakening of the independence axiom of Theorem 8 is *independence of never-approved alternatives*, which prescribes that $f_Y(P) = f_Z(P)$ for all agendas Y and Z and profiles P on Y such that $Z \subseteq Y$ and no voter in P approves any alternative in $Y \setminus Z$. Under dichotomous preferences, these alternatives are least-preferred by all voters, and are thus certainly Pareto dominated. As example Example 5 in Section 7 shows, AV is not the only rule satisfying this weaker independence condition together with the other axioms in Theorem 8. However, AV is characterized when adding an axiom often called reversal symmetry: if all voters switch to approving the complement of their ballot, then all chosen alternatives should become unchosen alternatives (unless all alternatives were chosen in the original profile). For a ballot $A \neq Y$, we denote by A^c the complement of A in Y, i.e., $A^c = Y \setminus A$; for $A = Y$, $A^c = Y$. Similarly, P^c is the profile where $P^c(A^c) = P(A)$ for all ballots $A \in \mathcal{A}_Y$. Then *reversal symmetry* requires that $f(P) \cap f(P^c) = \emptyset$ whenever $f(P) \neq X$. In the context of linear orders, these axioms characterize Borda's scoring rule (Morkelyunas, 1982, see also Saari and Barney, 2003).

Theorem 9. *AV is the only ballot aggregation function satisfying reinforcement, reversal symmetry, and independence of never-approved alternatives.*

Proof. Fix some agenda Y. We show that f_Y satisfies faithfulness and disjoint equality and then invoke Theorem 1 to conclude $f_Y = AV_Y$.

For faithfulness, fix some ballot A. Then independence of never approved alternatives implies $f(A) = f_A(A) = A$. For agenda A, $A = A^c$, and so $f_A(A) = f_A(A^c)$. Then reversal symmetry implies $f_A(A) = A$, which shows faithfulness.

For disjoint equality, let A, B be two disjoint ballots and $P = A + B$. Independence of never-approved alternatives implies $f(P) = f_{A\cup B}(P)$. For agenda $A \cup B$, $P = P^c$, and so $f_{A\cup B}(P) = f_{A\cup B}(P^c)$. Then reversal symmetry implies $f_{A\cup B}(P) = A \cup B$, which shows disjoint equality. □

7 Independence of Axioms

In each of our characterizations, the axioms used are independent: when dropping any one of the axioms, other rules satisfy the remaining axioms. To show this, we give a lengthy list of example rules below that satisfy particular combinations of axioms. Most of these examples are technical in nature, and not in themselves interesting. To see which example can be used to prove independence in a specific result, refer to Table 1.

Example 1. The rule that is like AV but counts voter 1 double.

This rule is not anonymous, but it satisfies neutrality, reinforcement, faithfulness, Fishburn-strategyproofness, clone consistency, reversal symmetry, independence of losers, of dominated alternatives, and of never-approved alternatives. That the rule satisfies these axioms is easily deduced from the fact that AV satisfies them.

Example 2. The rule PO selecting all Pareto optimal alternatives.

This rule fails reinforcement, since $\mathrm{PO}(a{+}c) = \{a, c\}$ and $\mathrm{PO}(b{+}c) = \{b, c\}$, but $\mathrm{PO}(a{+}b{+}c{+}c) = \{a, b, c\}$. However, PO satisfies anonymity, neutrality, disjoint equality, cancellation, clone consistency, independence of losers, and independence of dominated alternatives, and is not manipulable for Kelly's extension.

Example 3. The rule AV_{lex} selecting the lexicographically first approval winner.

This rule fails neutrality, but it satisfies anonymity, reinforcement (if a is the lexicographically first approval winner in both P and P', then $AV(P)$ and $AV(P')$ intersect, and so $AV(P + P') = AV(P) \cap AV(P')$ because AV satisfies reinforcement, and a is the lexicographically first element of the latter set), Fishburn-strategyproofness (since AV satisfies Fishburn-strategyproofness and lexicographic tie-braking is consistent across choice sets), independence of losers and of dominated alternatives (since AV satisfies these properties).

Example 4. The constant rule always selecting $\{x\}$ for some fixed $x \in X$.

This rule fails neutrality, but satisfies anonymity, reinforcement, and Fishburn-strategyproofness.

Example 5. The plurality rule ignoring all non-singleton ballots.

This rule is manipulable for Kelly's extension since $f(\{a\} + \{b, c\}) = \{a\}$ where the second voter can manipulate to $f(\{a\} + \{b\}) = \{a, b\}$. It fails majority consistency and fails to avoid Condorcet losers since $f(\{a\} + \{b, c\} + \{b, c\}) = \{a\}$. It fails clone consistency since $f(\{a\} + \{b\}) = \{a, b\}$ but $f(\{a\} + \{b, c\}) = \{a\}$. It fails independence of losers and of dominated alternatives, since $f(\{a\} + \{a, b\} + \{c\}) = \{a, c\}$ but $f(\{a\} + \{a\} + \{c\}) = \{a\}$. It fails reversal symmetry since $f(\{a\} + \{b, c\}) = \{a\}$ but $f(\{b, c\} + \{a\}) = \{a\}$. However, it is anonymous, neutral, and satisfies reinforcement and continuity because it is a scoring rule. It also satisfies independence of never-approved alternatives, since if we delete a never-approved alternative, the set of voters with a singleton ballot does not change.

Example 6. The rule CNL selecting all alternatives that are not Condorcet losers.

This rule fails reinforcement, since $f(\{a\} + \{b\}) = \{a, b\}$ and $f(\{a\} + \{c\}) = \{a, c\}$ but $f(\{a\} + \{a\} + \{b\} + \{c\}) = \{a, b, c\}$. The rule is neutral, it avoids Condorcet losers by definition, and it satisfies continuity (since if a is the Condorcet loser in profile P, then a is the Condorcet loser in $P' + kP$ for sufficiently large k, so $a \notin f(P' + kP)$).

Example 7. The rule selecting the approval winners with highest plurality score.

This rule fails continuity, since if $P = \{b\}$ and $P' = \{a\}{+}\{a, b\}{+}\{b, c\}$, then $f(P) = \{a\}$, but $f(P + kP') = \{b\}$ for all k. The rule is anonymous, neutral, it satisfies reinforcement (since it is a composite scoring rule), and it avoids Condorcet losers and is majority consistent (since it returns a subset of the approval winners).

Example 8. The rule selecting the approval winners, considering only those ballots that occur most frequently in the profile.

This rule fails reinforcement since $f(\{a, b\} + \{a, c\}) = \{a\}$ and $f(\{a, b\}) = \{a, b\}$, but $f(\{a, b\}, \{a, b\}, \{a, c\}) = \{a, b\}$. It is anonymous and neutral. It satisfies majority consistency, since a ballot that is reported by a majority is a (uniquely) most-frequent ballot. It satisfies continuity since for any profiles P and P', we have for large enough k that $f(P' + kP) = f(P)$, since the most-frequent ballots in P become the most-frequent ballots in $P' + kP$ for large enough k. It satisfies independence of never-approved alternatives, since removing a never-approved alternative does not change any approval sets, and does not change which ballots are most-frequent.

Example 9. The rule $-AV$ returning the alternatives with lowest approval score.

Example 10. The rule $TRIV$ returning all alternatives.

Example 11. The rule like AV, but counting plurality and veto ballots double.

This rule fails independence of never-approved alternatives, since $f_{\{a,b,c,d\}}(\{a,b\},\{c\}) = \{c\}$ but $f_{\{a,b,c\}}(\{a,b\},\{c\}) = \{a,b,c\}$. It is anonymous, satisfies reinforcement (since it is a scoring rule), and satisfies reversal symmetry (since AV satisfies reversal symmetry, and reversing a profile preserves voter weights).

Example 12. The rule like AV, but counting veto ballots double.

This rule fails disjoint equality and cancellation in 2-voter profile, since $f(\{a\} + X \setminus \{a\}) = X \setminus \{a\}$. It satisfies reinforcement (being a scoring rule), and faithfulness. It also satisfies cancellation in 3-voter profiles since if A, B, C are disjoint ballots, then none of A, B, C can be a veto ballot (we disallow empty ballots here).

Example 13. The scoring rule where a ballot A gives $|A|(|X| - |A|)$ points to each approved alternative.

This rule fails cancellation on 3-voter profiles because for $|X| \geqslant 4$, we have $f(\{a\} + \{b\} + X \setminus \{a,b\}) = X \setminus \{a,b\}$ because a and b have $|X| - 1$ points each and other alternatives have $(|X| - 2) \cdot 2 > |X| - 1$ points each. However, this rule satisfies reinforcement (since it is a scoring rule), faithfulness, and cancellation on 2-voter profiles (since if $A \cup B = X$ and $A \cap B = \emptyset$, then $|A|(|X| - |A|) = |B|(|X| - |B|)$, and so A and B give the same number of points to each approved alternative, and hence $f(A + B) = X$).

Example 14. The rule that is like AV, except that the ballot $\{a\}$ gives 1 point to a (as usual), and 0.5 points to b and c each.

This rule fails neutrality. Assume $|X| \geqslant 4$. The rule satisfies reinforcement and continuity (since it is a scoring rule). The rule also avoids Condorcet losers. Note that each alternative's score under f is at least that alternative's approval score, and they are equal except possibly for b and c. Let P be a profile with a Condorcet loser ℓ, where $\ell \neq b$ and $\ell \neq c$. Then ℓ's score is ℓ's approval score. Take $d \in X \setminus \{\ell, b, c\}$. Then d's score is d's approval score, so d's score is strictly higher than ℓ's score, since ℓ is the unique Condorcet loser. Hence $\ell \notin f(P)$. Let P be a profile where b is the Condorcet loser (in particular, c majority-beats b), yet $b \in f(P)$. Delete all ballots $\{a\}$ from P to obtain P'. In P', c still majority-beats b. Since f behaves like AV on P', c has strictly more points than b in P'. But since the $\{a\}$-ballots give the same number of points to b and c, c must have had strictly more points than b in P, contradicting that $b \in f(P)$; so $b \notin f(P)$. Symmetrically, we have $c \notin f(P)$ whenever c is the Condorcet loser in P.

Example 15. The rule where each voter gives $1 + \epsilon$ points to approved alternatives, and 1 point to a if a is not approved (where ϵ is infinitesimally small).

An alternative description of this rule is: if the ballots of all voters intersect (a consensus profile), then return that intersection. Otherwise, return $\{a\}$. This rule is not neutral, but it satisfies faithfulness. Being a composite scoring rule, it satisfies reinforcement. It is also Fishburn-strategyproof: in a consensus profile, for each voter, a subset of approved alternatives is selected, so the sincere ballot is a weakly dominant strategy for Fishburn's extension. If $\{a\}$ is returned, then a voter can only change this by approving alternatives that all other voters approve, but the new set of winners will be weakly dominated by $\{a\}$ for Fishburn's extension.

Example 16. The rule that is like AV, except that the ballot approving a gives 2 points to a and 1 point to every other approved alternative, and ballots disapproving a give -1 points to a and 1 point to each approved alternative.

This rule fails faithfulness since $f(\{a, b\}) = \{a\}$, and it fails neutrality by the same example. The rule is anonymous, satisfies reinforcement and continuity (since it is a scoring rule), disjoint equality (since for disjoint A and B, either neither approves a so $f(A+B) = A \cup B$, or exactly one of them approves a, whence a gets $2 + -1 = 1$ points, and every other alternative in $A \cup B$ gets 1 point as well, so $f(A + B) = A \cup B$), and majority consistency (suppose in profile P, ballot A is reported by more than half of the voters. If $a \notin A$, then each alternative in A has score at least $P(A)$ and all alternatives not in A have a lower score. If $a \in A$, then each alternative in A other than a has score at least $P(A)$, alternative a has score $2P(A) - \sum_{B \in \mathcal{A}:\, a \notin B} P(B) > P(A)$ because a minority disapproves a, and alternatives outside A have score less than $P(A)$. In either case $f(P) \subseteq A$).

Example 17. The rule where each ballot assigns $+1$ points to approved alternatives and -1 points to disapproved alternatives, except that ballots approving a but disapproving b give $+2$ points to a and -2 points to b, and that ballots approving b but disapproving a give $+2$ points to b and -2 points to a.

This rule fails faithfulness and fails neutrality, since $f(\{a, c\}) = \{a\}$. Since it is a scoring rule, it satisfies anonymity, reinforcement, and continuity. It satisfies cancellation, since in a profile in which all approval scores are equal, we have $P[a/b] = P[b/a]$ (in the notation of the proof of Theorem 1), and so under this rule all alternatives get 0 points in total.

8 Related Characterizations of Approval Voting

The literature on axiomatic characterizations of approval voting and variants thereof goes well beyond what we have discussed above.

Using variants of the axioms in Theorem 1, Sertel (1988) shows that AV is the only ballot aggregation function that satisfies anonymity, weak unanimity (faithfulness), weak consistency (reinforcement where one of the profiles is a single-voter profile), and strong disjoint equality. The latter property requires that if a ballot A contains none of the winners for a profile P, then the winners for the profile $P + A$ are the winners for P and alternatives in A whose approval score in P is one less than the maximal approval score.

Fishburn (1979) uses neutrality, continuity, and reinforcement to characterize the class of *scoring rules* on the domain of approval ballots. A scoring rule is specified by a vector $(s_1, \dots, s_m) \in \mathbb{R}^m$ assigning a score to each ballot cardinality. Given a profile P, the score of $x \in X$ is $\sum_{A \in \mathcal{A}:\, x \in A} s_{|A|} \cdot P(A)$. The scoring rule returns the set of alternatives with the highest score. For example, AV is the scoring rule $(1, \dots, 1)$; cumulative voting is the scoring rule $(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{m})$; and plurality voting is the scoring rule $(1, 0, \dots, 0)$ (which ignores all non-singleton ballots). Without the continuity axiom, Fishburn obtains the class of composite scoring rules, where each score s_i is specified by a vector in $\mathbb{R}^m$ (rather than a number in $\mathbb{R}$), and we compare vectors in $\mathbb{R}^m$ lexicographically. An example of such a rule is approval voting, with ties broken in favor of candidates who have the highest plurality score. Fishburn's result is closely related to Young's (1975) characterization of scoring rules when ballots are linear orders over alternatives, and the characterizations of scoring rules by Myerson (1995) and Pivato (2013) in an abstract setting. Notably, Fishburn's characterization (which uses a separating hyperplanes argument) works even on restricted domains where only certain cardinalities of approval ballot are allowed (e.g., approve at most three alternatives). In contrast, other characterizations of approval voting using direct proofs (including the ones in this paper) require that all cardinalities are allowed. An exception is a paper by Leach and Powers (2019) who show that the approval voting characterization of Alós-Ferrer (2006) holds also for many cardinality-restricted domains.

Theorem 10 of Fishburn (1979) shows that AV is the only rule for which sincerely reporting ones ballot is the only undominated strategy, where dominance is defined similarly to Kelly's extension. A related result by Vorsatz (2008) shows that AV is the only non-manipulable scoring rule on approval ballots for a notion of non-manipulability that is slightly weaker than Kelly's. Alcalde-Unzu and Vorsatz (2009) characterize the class of *size* AV rules, which are scoring rules where the score of a ballot weakly decreases in the number of approved alternatives. In addition to anonymity, neutrality, reinforcement, and a continuity axiom, they assume congruity (adding a voter who disapproves a losing alternative does not make it a winner) and contraction (removing alternatives from a voter's ballot does not add new winners unless all winners are removed).

Vorsatz (2007) considers ballot aggregation functions for variable sets of voters and alternatives. The choices for variable sets of alternatives have to be rationalizable by a transitive relation (cf. Sen, 1977) and indifferent voters do not influence the choice from two-alternatives sets. He shows that AV is the only such ballot aggregation function that is anonymous, neutral, non-manipulable (as in Vorsatz (2008)), and strictly monotonic, where the latter requires that ties on two alternative sets are resolved if one alternative gains additional support. Sato (2019) shows that this result even holds for a fixed set of voters, i.e., without the condition that relates different electorates to each other. Moreover, he characterizes approval voting using anonymity, neutrality, (a slight weakening of) strict monotonicity, and a stronger notion of non-manipulability. Neither of these conditions connects different agendas, so the result holds even for a fixed agenda.

In the framework of Vorsatz (2007), Massó and Vorsatz (2008) characterize *weighted* AV, where the approval score of an alternative is multiplied by its weight, which is exogenously given and fixed across ballot profiles. Their result requires anonymity, reinforcement, and two properties that put bounds on the ratios of the weights.

Sato (2014) and Alcalde-Unzu and Vorsatz (2014) characterize approval voting using a variable-agenda property stronger than some of the properties we have considered in Section 6: if the set of feasible alternatives shrinks, then we need to select precisely those previously winning alternatives that are still feasible, should any exist.

Baigent and Xu (1991) consider rules that aggregate choice functions into a collective choice function, i.e., a function that specifies the collective choice from each subset of alternatives. Here the approval score of an alternative corresponds to the number of voters who choose it. If the aggregation rule yields collective choice functions satisfying neutrality, positive responsiveness (additional support for a chosen alternative makes it the unique choice), and independence of symmetric substitutions (the collective choice only depends on the vector of approval scores), then it has to be approval voting.

9 Discussion

We have provided eight characterizations of approval voting based on the reinforcement axiom in combination with other appealing properties. These results are strong arguments for using approval voting once we accept the premise that voters have dichotomous preferences. Crucially, all our results hinge on reinforcement. In the context of political elections, making consistent choices across sub-electorates prevents severe instances of gerrymandering and is thus particularly important.

On a more technical note, many reinforcement-based results in social choice theory reason over an unbounded number of voters, e.g., when employing convex separation theorems on the set of fractional preference profiles. By contrast, some of our proofs allow us to derive bounds on the number of voters. For example, if a ballot aggregation function satisfies all assumptions of Theorem 1 on a set of n voters and all its subsets, then it has to be approval voting on subsets of up to $n/2$ voters. The proofs of Theorems 4 and 5 do not yield such bounds, since continuity requires arbitrarily large electorates.

The examples in Section 7 show that no axiom can be dropped from any of our characterizations. For Theorems 7 and 8, we use the ballot aggregation function choosing the lexicographically first approval winner, AV_{lex}, as an example that neutrality cannot be dropped. However, AV_{lex} is a refinement of AV, and thus identical to it whenever the approval winner is unique. We leave as an open question whether there is a ballot aggregation function satisfying all axioms of Theorems 7 or 8 except for neutrality, and that is not a subset of approval voting.

Lastly, in this paper we have considered the case of choosing a set of winning alternatives. However, approval-based rules are also natural candidates for choosing rankings, committees, or lotteries over alternatives. For example, Lackner and Skowron (2018) study approval-based rules for electing committees. Further study of other output types seems promising.

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APPENDIX

Lemma 1. *For a ballot aggregation function f, the following implications hold.*

(i) Reinforcement, disjoint equality, and neutrality imply faithfulness

(ii) Reinforcement, faithfulness, and cancellation imply disjoint equality

Proof. (i) Let A be a ballot. Neutrality implies that $f(A) \in \{X, A, X \setminus A\}$. If $A = X$, the first and the second option coincide and the third option is impossible, since choice sets have to be non-empty. Otherwise, assume for contradiction that $f(A) \in \{X, X \setminus A\}$ and let $b \in f(A) \setminus A$. If $A = \{a\}$ neutrality implies that $f(\{b\})$ is either $X \setminus \{b\}$ or X. Since $|X| \geqslant 3$, $f(\{a\}) \cap f(\{b\}) \neq \emptyset$ and thus reinforcement implies $f(\{a\}) \cap f(\{b\}) = f(\{a\} + \{b\})$, which is either $X \setminus \{a, b\}$ or X, both of which contradict disjoint equality.

For all remaining A, we have $f(A) \cap f(\{b\}) \neq \emptyset$, since $f(\{b\}) = \{b\}$ by the previous case. Then reinforcement implies $\{b\} = f(A) \cap f(\{b\}) = f(A + \{b\})$, which again contradicts disjoint equality.

(ii) By Theorem 1 and Remark 1, it suffices to show that $f(A + B) \supseteq A \cup B$ for all disjoint ballots A, B. Let $C = X \setminus (A \cup B)$. Cancellation implies that $f(A \cup B + C) = X$ and $f(A + B + C) = X$. So we have

$$\begin{aligned} f(A + B) &= f(A + B) \cap f(A \cup B + C) \\ &= f(A + B + A \cup B + C) \\ &= f(A \cup B) \cap f(A + B + C) = A \cup B, \end{aligned}$$

where the second and third equality follow from reinforcement and the last equality follows from faithfulness.

□

Lemma 1, in combination with Theorem 1, can be used to derive the characterizations of Alós-Ferrer (2006) and Fishburn (1978) as listed in Table 1. For the latter result, our proofs avoid the implicit use of anonymity by Fishburn (1978). In the proof of Lemma 1(ii), note that cancellation was only used for two and three voters with disjoint ballots (see also footnote 6). A related result was obtained by Ninjbat (2012) who showed that the only ballot aggregation functions satisfying reinforcement, neutrality, and cancellation are AV, $-AV$, and $TRIV$.

Theorem 3. *AV is the only non-trivial ballot aggregation function satisfying reinforcement, continuity, and semi-Condorcet consistency.*

Proof. The proof proceeds in three stages: first we show that Condorcet winners have to be chosen uniquely whenever they exist; second that only approval winners can be chosen; and third that exactly the approval winners are chosen.

We start by constructing a relation $E \subseteq X \times X$ on alternatives, where $(a,b) \in E$ if for all profiles P, $a \in f(P)$ implies $b \in f(P)$. The following claim will turn out useful to derive properties of this relation:

Claim 1. If b is a Condorcet winner in a profile P and $a \in f(P)$, then $(a,b) \in E$.

Proof of Claim 1. Assume for contradiction that $(a,b) \notin E$. By definition of E, there is profile P' such that $a \in f(P')$ and $b \notin f(P')$. Since b is a Condorcet winner in P, we can find an integer k large enough so that b is a Condorcet winner in the profile $P' + kP$. (For example, any k larger than the number of voters in P' works.) Semi-Condorcet consistency of f then gives $b \in f(P' + kP)$. By reinforcement, we have $f(kP) = f(P)$ and so $a \in f(P') \cap f(kP)$. Thus we can apply reinforcement again to get

$$f(P') \cap f(kP) = f(P' + kP).$$

From this we get $b \notin f(P' + kP)$, since $b \notin f(P')$, which is a contradiction. □

Now we deduce some properties of E. Clearly, it is reflexive and, since logical implication is transitive, it is transitive. Moreover, E is symmetric. Suppose that $(a,b) \in E$. Let P be a profile where a is a Condorcet winner. Since f is semi-Condorcet consistent, $a \in f(P)$, and since $(a,b) \in E$, $b \in f(P)$. Then by Claim 1, $(b,a) \in E$. So E is an equivalence relation and partitions alternatives into equivalence classes.

To proceed, we distinguish three cases. If $E = X \times X$, it puts all alternatives into the same equivalence class and f is the trivial rule: choice sets are non-empty, and if any alternative is in the choice set, all alternatives have to be chosen by definition of E. This contradicts our assumption that f is non-trivial.

If $E = \{(a,a) \colon a \in X\}$, then by Claim 1, Condorcet winners have to be chosen uniquely whenever they exist.

In all other cases, there are alternatives a, b, c such that a and b belong to the same equivalence class and c belongs to a different one. In particular, a is chosen whenever b is chosen and *vice versa*. Moreover, if either a or b is a Condorcet winner in a profile P, then c is not chosen. Otherwise we would have $(c,a) \in E$ or $(c,b) \in E$ by Claim 1, which cannot be since c is in a different equivalence class than a and b. Thus, for the profiles $P = 3\{a\} + 2\{c\}$ and $P' = 3\{b\} + 2\{c\}$, the following hold: $a \in f(P)$ since f is semi-Condorcet consistent; thus $b \in f(P)$ and $c \notin f(P)$; likewise, $a, b \in f(P')$ and $c \notin f(P')$. Then by reinforcement, $f(P + P') = f(3\{a\} + 3\{b\} + 4\{c\}) = f(P) \cap f(P')$ and so $a, b \in f(P + P')$ and $c \notin f(P + P')$. However, c is a Condorcet winner in $P + P'$, contradicting semi-Condorcet consistency of f. So we conclude that only the second case is possible, i.e., Condorcet winners have to be chosen uniquely whenever they exist.

Second we show that f has to choose a subset of approval winners for all profiles. Assume for contradiction that for some profile P, $f(P)$ contains an alternative a that is not in $AV(P)$. Let $b \in AV(P)$ and $P' = 2\{a\} + \{b\}$. Then b is a Condorcet winner in the profile $2P + P'$, since $(2P+P')[b] = 2P[b]+1$, $(2P+P')[a] = 2P[a]+2 \leqslant 2(P[b]-1)+2 = 2P[b]$, and $(2P + P')[c] = 2P[c] \leqslant 2P[b]$ for all $c \neq a, b$, since $b \in AV(P)$. Recall that Condorcet winners have to be chosen uniquely whenever they exist. So $f(2P+P') = \{b\}$. Moreover, a is a Condorcet winner in the profile P' and thus $f(P') = \{a\}$. As observed earlier, reinforcement implies that $f(P) = f(2P)$. Since $a \in f(P)$ by assumption and $f(P') = \{a\}$, reinforcement thus implies $f(2P + P') = f(2P) \cap f(P') = f(P) \cap f(P') = \{a\}$, which is a contradiction. We conclude that $f(P) \subseteq AV(P)$ for all profiles P.

Third we show that $f(P) = AV(P)$ for all P. We have already solved the case of a unique approval winner in the first part. If there are exactly two approval winners in P and f only chooses one of them, say $AV(P) = \{a, b\}$ and $f(P) = \{a\}$, then by continuity, $f(\{b\} + kP) = \{a\}$ for some integer k. But b is a Condorcet winner in the profile $\{b\} + kP$ and so $f(\{b\} + kP) = \{b\}$, which is a contradiction. Now, for arbitrary P, assume for contradiction that $a \in f(P) \subseteq AV(P)$ and $b \in AV(P) \setminus f(P)$. From the case of two approval winners we know that $f(\{a, b\}) = \{a, b\}$. In the profile $P + \{a, b\}$ only a and b are approval winners, and so $f(P + \{a, b\}) = AV(P + \{a, b\}) = \{a, b\}$. Applying reinforcement we get

$$f(P + \{a, b\}) = f(P) \cap f(\{a, b\}) = \{a\},$$

since $b \notin f(P)$, which is a contradiction. We conclude that $f(P) = AV(P)$. □

Theorem 5. *AV is the only non-trivial ballot aggregation function satisfying reinforcement, neutrality, continuity, and majority consistency.*

Proof. We show that any such ballot aggregation function satisfies faithfulness and disjoint equality and then apply Theorem 1. First we prove faithfulness. Let A be some ballot. Neutrality and majority consistency imply that $f(A) \in \{X, A\}$. If $f(A) = X$ for all ballots A, then by reinforcement, $f = TRIV$, which is contrary to the assumption. So there is some ballot A such that $f(A) = A$. For $c \in X \setminus A$, we have that $f(\{c\}) \in \{X, \{c\}\}$. If $f(\{c\}) = X$, then reinforcement implies that $f(A + 2\{c\}) = A$, which contradicts majority consistency. So by neutrality, $f(\{c\}) = \{c\}$ for all alternatives c. Now let A be an arbitrary ballot and assume for contradiction that $f(A) = X$. For $c \in X \setminus A$, we have $f(\{c\}) = \{c\}$ and so reinforcement implies $f(2A + \{c\}) = \{c\}$, which contradicts majority consistency. Thus, $f(A) = A$ and f satisfies faithfulness.

Second we prove disjoint equality. Let A, B be two disjoint ballots. Neutrality implies that $f(A+B) \in \{X, A, B, A\cup B, X\setminus A, X\setminus B, X\setminus(A\cup B)\}$. We will exclude all possibilities except for $f(A+B) = A\cup B$. Assume for contradiction that $f(A+B)$ contains some $c \in X\setminus(A\cup B)$. Moreover, let $a \in A$. Faithfulness implies that $f(A\setminus\{a\}\cup\{c\}) = A\setminus\{a\}\cup\{c\}$. Reinforcement then implies $f(A + B + A \setminus \{a\} \cup \{c\}) = f(A + B) \cap (A \setminus \{a\} \cup \{c\})$. In particular, $c \in f(A + B + A \setminus \{a\} \cup \{c\})$ and $a \notin f(A + B + A \setminus \{a\} \cup \{c\})$. This contradicts neutrality, since the profile $A + B + A \setminus \{a\} \cup \{c\}$ is symmetric with respect to a and c, so that either both of them or neither of them has to be chosen. Thus, $f(A+B) \in \{A\cup B, A, B\}$. The remainder of the proof excludes the latter two possibilities. It proceeds along three increasingly general cases.

Case 1: $A = \{a\}$ and $B = \{b\}$ for two alternatives a, b. By the previous analysis and neutrality, remains $f(\{a\} + \{b\}) = \{a, b\}$ as the only possibility.

Case 2: Arbitrary cardinality A and $B = \{b\}$ for some alternative b. If $f(A + \{b\}) = \{b\}$, continuity implies that $f(A+k(A+\{b\})) = \{b\}$ for some integer k, which contradicts majority consistency. If $f(A+\{b\}) = A$, reinforcement and Case 1 imply that for $a \in A$,

$f(A + \{b\} + \{a\} + \{b\}) = \{a\}$. Then, continuity implies that $f(\{b\} + k(A + \{b\} + \{a\} + \{b\})) = \{a\}$ for some integer k, which contradicts majority consistency. Thus, $f(A + \{b\}) = A \cup \{b\}$.

Case 3: Arbitrary cardinality A, B. If $f(A + B) = B$, choose some $b \in B$. Case 2 implies that $f(A + \{b\}) = A \cup \{b\}$. So $f(A + B + A + \{b\}) = \{b\}$. Then continuity implies that $f(A + k(A + B + A + \{b\})) = \{b\}$ for some k, which contradicts majority consistency. The case $f(A + B) = A$ is analogous. Thus, $f(A + B) = A \cup B$ remains as the only possibility. □

Theorem 6. *AV is the only ballot aggregation function satisfying reinforcement, clone consistency, and faithfulness if* $|X| \geqslant 4$.

Proof. Take any such ballot aggregation function f and some agenda $Y \in \mathcal{P}(X)$. We will omit the subscript Y for f within this proof. We show that f satisfies disjoint equality and then apply Theorem 1.

Let $A, B \in \mathcal{A}_Y$ be two disjoint ballots and $C = Y \setminus (A \cup B)$. In the two-voter profile $A+B$ all alternatives in A are clones of each other and likewise all alternatives in B and in C. So clone consistency implies that for each of these sets, either all alternatives are chosen or none, i.e., $f(A + B) \in \{Y, A \cup B, A, B, C\}$.

We show that in addition, A is chosen if and only if B is chosen. So assume that $A \subseteq f(A + B)$, let $a \in A$, $b \in B$, and $c \in C$, and consider the agenda $\{a, b, c\}$. (If $A \cup B = Y$, omit c in what follows.) Clone consistency implies that $a \in f_{\{a,b,c\}}(A + B) = f(A + B) \cap \{a, b, c\}$. Let $x \in X \setminus \{a, b, c\}$, which exists since $|X| \geqslant 4$, and consider the profile $\{a, x\} + \{b\}$ on the agenda $\{a, b, c, x\}$. Clone consistency implies $x \in f_{\{a,b,c,x\}}(\{a, x\} + \{b\})$. Further applications of clone consistency imply that $x \in f_{\{b,c,x\}}(\{x\}+\{b\})$, $x \in f_{\{a,b,c,x\}}(\{x\}+\{a, b\})$, $x \in f_{\{a,c,x\}}(\{x\}+\{a\})$, $b \in f_{\{a,b,c,x\}}(\{b, x\}+\{a\})$, and $b \in f_{\{a,b,c\}}(\{b\} + \{a\})$. So we get $b \in f_{\{a,b,c\}}(A + B) = f(A + B) \cap \{a, b, c\}$ and thus, $B \subseteq f(A + B)$.

The remaining possibilities are $f(A + B) \in \{Y, A \cup B, C\}$. Assume for contradiction that $f(A+B) \cap C \neq \emptyset$. Faithfulness implies $f(C) = C$. Applying reinforcement to $A+B$ and the one-voter profile C yields $f(A + B + C) = f(A + B) \cap f(C) = C$. Essentially the same line of reasoning as in the previous paragraph shows that $f(A + B + C) = Y$, which is a contradiction. Thus $f(A+B) = A \cup B$, and so f satisfies disjoint equality. □

Theorem 7. *AV is the only ballot aggregation function satisfying reinforcement, neutrality, faithfulness, and independence of losers.*

Proof. Take any such ballot aggregation function f and some agenda $Y \in \mathcal{P}(X)$. We will omit the subscript Y for f within this proof. We show that f satisfies disjoint equality and then apply Theorem 1.

To this end, let A, B be two disjoint ballots and $a \in A$ and $b \in B$. Neutrality implies that $f(A + B) \in \{X, A \cup B, A, B, X \setminus B, X \setminus A\}$. First assume for contradiction that $f(A+B) \not\subseteq A \cup B$ and let $c \in f(A+B) \setminus (A \cup B)$. By faithfulness, we have $f(\{c\}) = \{c\}$. Hence, by reinforcement, $f(A + B + \{c\}) = \{c\}$. Independence of losers implies that

$f(A+B+\{c\}) = f_{\{a,b,c\}}(\{a\}+\{b\}+\{c\}) = \{c\}$, which contradicts neutrality. So we have $f(A+B) \in \{A \cup B, A, B\}$.

Second, consider the case $f(A+B) = A$. From the previous case and neutrality, we know that $f(\{a\}+\{b\}) = \{a,b\}$. Hence, by reinforcement, $f(A+B+\{a\}+\{b\}) = \{a\}$. Independence of losers implies that $f(A+B+\{a\}+\{b\}) = f_{\{a,b\}}(\{a\}+\{b\}+\{a\}+\{b\}) = \{a\}$, which again contradicts neutrality. Similarly, we get a contradiction if $f(A+B) = B$. Hence, $f(A+B) = A \cup B$ remains as the only possibility and so f satisfies disjoint equality as desired. □

Theorem 8. *AV is the only ballot aggregation function satisfying reinforcement, neutrality, and independence of Pareto dominated alternatives.*

Proof. Fix some agenda Y. We show that f_Y satisfies disjoint equality and then invoke Fishburn's (1978) result to conclude $f_Y = AV_Y$. The subscript Y will be omitted in the rest of the proof. Let A, B be two disjoint ballots. Observe that in the profile $A+B$ all alternatives in $X \setminus (A \cup B)$ are Pareto dominated by alternatives in $A \cup B$. So independence of Pareto dominated alternatives implies $f(A+B) = f_{A \cup B}(A+B) \subseteq A \cup B$. Then it follows from neutrality that $f(A+B) \in \{A, B, A \cup B\}$. Without loss of generality, we may assume for contradiction that $f(A+B) = A$. For $a \in A$ and $b \in B$, neutrality and independence of Pareto dominated alternatives imply $f(\{a\}+\{b\}) = f_{\{a,b\}}(\{a\}+\{b\}) = \{a,b\}$. Then it follows from reinforcement that $f(A+B+\{a\}+\{b\}) = \{a\}$. In the profile $A+B+\{a\}+\{b\}$, all alternatives except a and b are Pareto dominated by either a or b. Thus, by independence of Pareto dominated alternatives, $\{a\} = f(A+B+\{a\}+\{b\}) = f_{\{a,b\}}(\{a\}+\{b\}+\{a\}+\{b\})$, which contradicts neutrality. □